

# Quantization of Systems Reduced by Commuting Hamiltonian Flows, a Decomposable Weyl Calculus and Commutation of Quantization and Reduction.

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## 1 Introduction

We claim that the following constructions in Classical and Quantum Mechanics are in some sense analogue. We will describe their common features and clarify under what conditions the similarities occur.

Let  $\Sigma$  be a real symplectic manifold,  $h_1, \dots, h_k \in C^\infty(\Sigma)$  be a finite family of pairwise Poisson-commuting complete real Hamiltonians and denote by  $\Phi^1, \dots, \Phi^k$  their respective flows.

Also let  $J = (h_1, \dots, h_k)$  and  $\Phi_{t_1, \dots, t_k} := \Phi_{t_1}^1 \circ \dots \circ \Phi_{t_k}^k$ . Then, for each regular  $\lambda \in J(\mathbb{R}^{2n})$ , the energy level submanifold  $\hat{\Sigma}_\lambda := J^{-1}(\lambda)$  is invariant under  $\Phi$ , and it turns out that, under suitable assumptions, the orbits space  $\Sigma_\lambda := \hat{\Sigma}_\lambda / \Phi$  is a symplectic manifold.

On the quantum side, let  $H_1, \dots, H_k$  be a finite family of pairwise commuting selfadjoint operators on a Hilbert space  $\mathcal{H}$  and  $\text{sp}(H_1, \dots, H_k)$  its joint spectrum. Then there is a measure  $\eta$  on  $\text{sp}(H_1, \dots, H_k)$ , an  $\eta$ -mesurable field of Hilbert spaces  $\{\text{sp}(H_1, \dots, H_k) \ni \lambda \rightarrow \mathcal{H}(\lambda)\}$ , and a unitary operator  $T : \mathcal{H} \rightarrow \int_{\text{sp}(H_1, \dots, H_k)}^\oplus \mathcal{H}(\lambda) d\eta(\lambda)$  such that

$$(TH_j u)(\lambda) = \lambda_j(Tu)(\lambda) \quad \forall u \in \text{Dom}(H_j).$$

Already we can notice some of the similarities of both constructions: on each fiber each Hamiltonian is constant.

We claim the following property for the quantization of the Poisson-commutant of the family  $h_1, \dots, h_k$ : if  $h_j$  is quantized as  $H_j$  and  $\lambda \in J(\mathbb{R}^{2n}) \cap \text{sp}(H_1, \dots, H_k)$  is regular, then  $C^\infty(\Sigma_\lambda)$  should be quantized on  $\mathcal{H}(\lambda)$ ; therefore if  $J(\mathbb{R}^{2n}) \setminus \mathcal{I} \subseteq \text{sp}(H_1, \dots, H_k)$  a.e., then we can quantize the Poisson algebra  $\mathcal{A} := \{f \in C^\infty(\Sigma) : \{h_j, f\} = 0\}$  as selfadjoint operators on  $\mathcal{H}$  commuting with each  $H_j$ , where  $\mathcal{I}$  is the set of singular values of  $J$ .

On this article by quantization of a Poisson algebra  $\mathcal{P}$  on a Hilbert space  $\mathcal{H}$ , we mean a family of linear and injective maps  $\mathfrak{Op}_\hbar^\mathcal{P}$ , that associate to certain elements of  $\mathcal{P}$  a selfadjoint operator on  $\mathcal{H}$ , obeying semiclassical properties, where the index  $\hbar$  belongs to a subset of  $\mathbb{R}$  having 0 as an accumulation point. Of course this definition needs to be made more precise. One alternative is to require that the operators involved are bounded, this leads to the notion of strict deformation quantization [30]. But, since many unbounded operators appear naturally in Quantum Mechanics,

we prefer to work with the current notion of quantization. After that, we should restrict it to a subalgebra where we would obtain bounded operator.

Let us be more precise about our formal claim. The classical construction described above is a particular case of what is known as Marsden-Weinstein reduction [19]. Some generalizations of this construction were given in [35] and [14].

The key idea to justify our claim is to notice that Marsden-Weinstein reduction and its generalizations can be interpreted as procedures to induce representations of certain Poisson algebra. In our case the Poisson algebra is  $\mathcal{A}$  and it can be represented in each  $C^\infty(\Sigma_\lambda)$ . Then the quantum analogue of this procedure should do the same. This is precisely what Rieffel induction does. This procedure induce representations on any  $C^*$ -algebra of adjointable operators of a Hilbert  $\mathfrak{B}$ -module from representations of  $\mathfrak{B}$  (where  $\mathfrak{B}$  is also a  $C^*$ -algebra). So, we claim that  $\Sigma_\lambda$  should be quantized on a Hilbert space constructed through Rieffel induction. This ideas were first mentioned in [14, 15]. We also claim that  $\mathcal{A}$  should be quantized as an algebra of adjointable operators of the required Hilbert  $C^*$ -module.

The problem is to look for a  $C^*$ -algebra  $\mathfrak{B}$  and a Hilbert  $\mathfrak{B}$ -module that would represent the quantum analogue of the classical initial data. We propose to take  $\mathfrak{B} = C_0(\text{sp}(H_1, \dots, H_k))$ , assume that the measurable field of Hilbert spaces described before comes from a continuous one, and consider the Hilbert  $\mathfrak{B}$ -module canonically associated to such field. We will prove that the algebra of adjointable operators coincides with the algebra of decomposable operators; note that up to conjugate with  $T$ , this is the algebra of bounded operators commuting with each  $H_j$ .

Moreover, the Hilbert space constructed through Rieffel induction from the evaluation at  $\lambda$  representation is precisely  $\mathcal{H}(\lambda)$ . A sort of group-like approach for a solution to the problem of finding the right  $C^*$ -algebra and module, can be found in [15], though certain technical issues were omitted there. If we assume that  $\eta$  is absolute continuous respect to the Lebesgue measure then this approach also leads to the first part of our claim.

We will analyze the details of the justification above in section 2.

Assume that for  $\eta$ -almost every  $\lambda \in J(\Sigma) \setminus \mathcal{I}$ , we have a quantization  $\mathfrak{Op}_h^\lambda$  of  $C^\infty(\Sigma_\lambda)$  on  $\mathcal{H}(\lambda)$ . If  $f \in \mathcal{A}$  then  $f \circ \Phi_t = f$  for every  $t \in \mathbb{R}^k$ ; so for each  $\lambda \in J(\Sigma) \setminus \mathcal{I}$ , we can consider the smooth function  $f^\lambda$  on the orbit space  $\Sigma_\lambda$ , given by  $f^\lambda([\sigma]) = f(\sigma)$ , where  $[\sigma]$  denote the orbit of  $\sigma \in \hat{\Sigma}_\lambda$ . Then, if  $\mathfrak{Op}_h^\lambda$  is defined on  $f^\lambda$  for almost every  $\lambda$ , we can essentially define the field of operators  $\{J(\Sigma) \ni \lambda \rightarrow \mathfrak{Op}_h^\lambda(f^\lambda)\}$ . Therefore we can consider the operator on the Hilbert subspace  $\int_{J(\Sigma) \setminus \mathcal{I}}^\oplus \mathcal{H}(\lambda) d\eta(\lambda)$  denoted by  $\int_{J(\Sigma) \setminus \mathcal{I}}^\oplus \mathfrak{Op}_h^\lambda(f^\lambda) d\eta(\lambda)$  and defined fiberwise on a suitable domain. We then define a decomposable Weyl Calculus  $\mathfrak{Op}_h^d$  as:

$$\mathfrak{Op}_h^d(f) := T^* \left[ \int_{J(\Sigma) \setminus \mathcal{I}}^\oplus \mathfrak{Op}_h^\lambda(f^\lambda) d\eta(\lambda) \right] T;$$

The minimal requirement to obtain that  $\mathfrak{Op}_h^d$  is injective is that  $J(\Sigma) \setminus \mathcal{I} \subseteq \text{sp}(H_1, \dots, H_k)$ .

On such generality we are not able to say much more about  $\mathfrak{Op}_h^d$ , because it is not clear where it is defined, neither what is the domain of the operators defined above, or if some of them are selfadjoint; so we will consider some particular but important cases where we can deal with such issues. However, if  $\mathfrak{Op}_h^d(f)$  is selfadjoint, then it commutes with each  $H_j$ .

Consider  $\Sigma = \mathbb{R}^{2n}$  endowed with the canonical symplectic form and  $h_j(x, \xi) = \phi_j(x)$ , where each  $\phi_j \in C^\infty(\mathbb{R}^n)$ . Also let  $\tilde{J} = (\phi_1, \dots, \phi_k)$  and  $\tilde{\Sigma}_\lambda = (\tilde{J})^{-1}(\lambda)$ . Then  $\hat{\Sigma}_\lambda = \tilde{\Sigma}_\lambda \times \mathbb{R}^n$  and we will prove that  $\Sigma_\lambda \cong T^*\tilde{\Sigma}_\lambda$ . On the quantum side, we quantize each  $h_j$  as the multiplication operator given

by  $\phi_j$ . Then  $\text{sp}(H_1, \dots, H_k) = \overline{\tilde{J}(\mathbb{R}^n)}$ . We will prove that  $\eta(A) = m(A \cap \tilde{J}(\mathbb{R}^n))$ , where  $m$  is the Lebesgue measure. Morse-Sard Theorem implies  $\mathcal{I}$  has null Lebesgue measure, and we will show that if we assume  $\mathcal{I}$  is also closed, then we can make  $\{\tilde{J}(\mathbb{R}^n) \setminus \mathcal{I} \ni \lambda \rightarrow L^2(\tilde{\Sigma}_\lambda)\}$  a continuous field of Hilbert spaces. Finally, using co-area formula, we will find  $T$  explicitly with values in the direct integral of  $L^2(\tilde{\Sigma}_\lambda)$  respect to  $\eta$ . Essentially the same can be done if we take  $h_j(x, \xi) = \phi_j(\xi)$ ; we will have that  $\Sigma_\lambda \cong T^*\tilde{\Sigma}_\lambda$ , but endowed with minus the canonical symplectic form, then  $T$  become the same than in the previous case but composed with the Fourier transform.

The problem of quantizing the cotangent bundle  $T^*M$  on  $L^2(M)$ , where  $M$  is a Riemannian manifold, was already considered in [16] II.3, and a solution was given, which we denote by  $\mathfrak{Op}_\hbar^M$ . It define a strict deformation quantization on a suitable Poisson subalgebra of  $C^\infty(T^*M)$ .

Among the Hamiltonians over which it is defined, an important roles will be played by those that are only position dependents and by those of the form  $J_X(m, \xi) = \langle X(m), \xi \rangle_m$ , where  $X$  is a complete vector field on  $M$  and  $\langle \cdot, \cdot \rangle_m$  denotes the duality between the tangent and the cotangent space at the point  $m$ .  $\mathfrak{Op}_\hbar^M$  sends the first ones to the corresponding multiplication operators and it turns out that,

$$\mathfrak{Op}_\hbar^M(J_X) = -i\hbar(X + \frac{1}{2}\text{div}X).$$

When  $M = \tilde{\Sigma}_\lambda$ , we will denote this quantization by  $\mathfrak{Op}_\hbar^\lambda$ . We use it to define a decomposable Weyl Calculus  $\mathfrak{Op}_\hbar^d$  as above.

The goal for the remainder of this article is to justify why it is interesting to study  $\mathfrak{Op}_\hbar^d$ . For instance, we will show that, if  $\mathfrak{Op}_\hbar^d$  restricted to a suitable Poisson subalgebra of  $\mathcal{A}$  define bounded operators, we would get at least a weak version of strictness.

Note that the usual Weyl Calculus  $\mathfrak{Op}_\hbar$  does quantize  $\mathcal{A}$ , so one could wonder why it might be interesting to introduce  $\mathfrak{Op}_\hbar^d$ . The main advantage is that  $\mathfrak{Op}_\hbar$  define operators commuting with each  $H_j$ . Informally  $\mathfrak{Op}_\hbar$  does not consider the way  $\mathcal{A}$  is defined, because it does not intertwine the Poisson bracket with the commutator, but it does so semiclassically. So, one could expect that at least when  $\hbar \rightarrow 0$ ,  $\mathfrak{Op}_\hbar^d$  and  $\mathfrak{Op}_\hbar$  coincide. Formally, this is what we call semiclassical commutation of quantization and reduction.

It can happen that both calculus actually coincide on some elements of  $\mathcal{A}$ , we call this strong commutation of quantization and reduction. Of course this happens for only position dependent Hamiltonians. We will prove that this is also the case for Hamiltonians of the form  $J_X$ , with  $X$  a complete vector field on  $\mathbb{R}^n$  tangent to each  $\tilde{\Sigma}_\lambda$ .

Someone might wonder if  $\mathcal{A}$  is interesting enough to justify such effort. We will show that if a Lie group  $G$  acts on  $\mathbb{R}^n$  with infinitesimal generators tangent to each  $\tilde{\Sigma}_\lambda$ , then we can embed the Lie-Poisson algebra  $C^\infty(\mathfrak{g}^*)$  on  $\mathcal{A}$ . We will give an example of this as well.

Details of the corresponding decomposable Weyl Calculus obtained in these cases, commutation of quantization and reduction, and the embedding of Lie-Poisson algebras will be given in section 5.

The case  $k = 1$  and  $h_1(x, \xi) = |\xi|^2/2$  is of special interest for Physics applications, it is also interesting because of the geometry of  $\tilde{\Sigma}_\lambda = \mathbb{S}_{\sqrt{2\lambda}}^{n-1}$  (the  $(n-1)$ -sphere of radius  $\sqrt{2\lambda}$ ). In this case it is easy to compute explicitly  $\mathfrak{Op}_\hbar^\lambda$ . Also, since stereographic coordinates are normal, we think it might be useful to express  $\mathfrak{Op}_\hbar^\lambda$  using them, and we did so.

It is clear that the action of the orthogonal group  $O(n)$  on  $\mathbb{R}^n$  has infinitesimal generators tangent to each sphere, so we can embed  $C^\infty(\mathfrak{so}(n)^*)$  in  $\mathcal{A}$ . In particular, angular momenta belong to  $\mathcal{A}$ . At first we expected that, due to the standard proof of Groenwold-van Hove's Theorem, even though quantization and reduction commute strongly on certain  $J_X$ , it wouldn't be the case on some not

so large powers of them; however we checked that, the operators obtained by applying  $\mathfrak{Op}_\hbar$  to the  $m$ -powers of angular momenta with  $m = 1, 2, 3, 4$  still commute with the operator  $\mathfrak{Op}_\hbar(h_1)$ ; thus we provided evidence that quantization and reduction might commute strongly on them as well.

## 2 Quantizing the construction of $\Sigma_\lambda$ to obtain $\mathcal{H}(\lambda)$

Recall that  $\Sigma_\lambda$  is constructed by applying Marsden-Weinstein reduction. One of the key ideas in what follows is due to K. Landsman, who noticed that the quantum counterpart of his generalization of Marsden-Weinstein reduction should be Rieffel induction.

The purpose of this section is to quantize each element in our particular reduction to obtain a Hilbert  $C^*$ -module, identify its algebra of adjointable operators and finally apply Rieffel induction. We compared in the last subsection our viewpoint with another also due to K. Landsman. Before this, we will describe briefly Marsden-Weinstein reduction and Rieffel induction to explain Landsman's claim.

Although in the next section we gave several cases for which we showed that our claim is clear and the rest of the article is devoted to study them, we think that it could be important to understand our approach, which might support future works too. However, this section is not required in the remainder of the article, so the reader not interested can skip it.

### 2.1 Generalized Marsden-Weinstein reduction and Rieffel induction as its quantum counterpart

This subsection is devoted to describe briefly Marsden-Weinstein reduction, its generalization and to recall why Rieffel induction can be considered as its quantum counterpart.

Let  $(\Sigma, \omega)$  be a symplectic manifold equipped with a right action of a Lie group  $G$  by symplectic diffeomorphisms and let  $J : \Sigma \rightarrow (\mathfrak{g}^*)^-$  be a equivariant moment map. Here  $\mathfrak{g}$  denote the Lie algebra of  $G$  and we use the upper minus sign to emphasize that the dual  $\mathfrak{g}^*$  is endowed with minus the standard Lie-Poisson bracket. By equivariant we mean that  $J$  intertwines the action of  $G$  on  $\Sigma$  with the coadjoint action of  $G$  on  $\mathfrak{g}^*$ .

Let  $\mathcal{O} \subset \mathfrak{g}^*$  be a orbit for the coadjoint action. Then  $J^{-1}(\mathcal{O})$  is stable by the action of  $G$ , and if  $J$  is not singular then  $\Sigma_{\mathcal{O}} = J^{-1}(\mathcal{O})/G$  has a symplectic manifold structure. The above superficially described process is called Marsden-Weinstein reduction; for details on this topic see [2] or [18].

The space of smooth functions on  $\Sigma$  that are  $G$ -invariant on  $J^{-1}(\mathcal{O})$  forms a Poisson subalgebra of  $C^\infty(\Sigma)$ , which we denote by  $\mathcal{A}_{\mathcal{O}}$ . The quotient map  $J^{-1}(\mathcal{O}) \rightarrow \Sigma_{\mathcal{O}}$  is a Poisson map.

For our case we consider  $G = \mathbb{R}^k$  and  $J = (h_1, \dots, h_k)$  (the corresponding Lie algebra is just  $\mathbb{R}^k$  with trivial Lie bracket, and its dual is again  $\mathbb{R}^k$  with trivial Poisson manifold structure). Note that the coadjoint action in  $\mathbb{R}^k$  is also trivial, so we recover what was explained in the introduction. In general, given symplectic manifold  $\Sigma$ , a Poisson manifold  $P$  and a Poisson map  $J : \Sigma \rightarrow P^-$ , one can construct from each symplectic manifold  $\Sigma_\rho$  and a Poisson map  $\rho : \Sigma_\rho \rightarrow P^+$ , a new symplectic manifold  $\Sigma^\rho$  and a Poisson map  $ind(\rho)$  from

$$\mathcal{A}_\rho := \{f \in C^\infty(\Sigma)/\{f, g\}|_{J^{-1}[\rho(\Sigma_\rho)]} = 0 \ \forall g \in J^*[C^\infty(P)]\}$$

to  $C^\infty(\Sigma^\rho)$ . Therefore under such assumptions, for each  $\rho$  we get a Poisson map from  $\mathcal{A} := (J^*[C^\infty(P)])'$  to  $C^\infty(\Sigma^\rho)$  (we prefer to call Poisson maps both the linear maps between Poisson

algebras that intertwine brackets and smooth maps between Poisson manifolds with pullback satisfying that property). We call this procedure Landsman-Xu generalization. For a proper treatment of this subject see [14],[15] or [16].

Marsden-Weinstein reduction follows by noticing that the symplectic leaves of  $(\mathfrak{g}^*)^+$  are precisely the orbits of the coadjoint action, so the required Poisson map in this case is just the inclusion map. Formally one can think of Poisson algebras as abstract algebras of classical observables, while Poisson subalgebras of  $C^\infty(\Sigma')$ , for some symplectic manifold  $\Sigma'$ , as concrete algebras of classical observables. So a Poisson map from a Poisson algebra  $B$  to  $C^\infty(\Sigma')$  can be interpreted as a representation of  $B$ . Since  $\rho : \Sigma_\rho \rightarrow P^+$  is a Poisson map, Landsman-Xu generalization can be considered as a procedure meant to induce representations of the Poisson algebra  $\mathcal{A}$  from representations of  $C^\infty(P)$ . This is precisely what Rieffel induction does in the  $C^*$ -algebraic (which we interpret as quantum) framework: it induces representations of a  $C^*$ -algebra from representations of another one. This is one of the main reason to consider it as the quantum counterpart of the Landsman-Xu generalization. One can find in [14] and [16] a list of extra justifications to consider Rieffel induction as a generalized quantum Marsden-Weinstein reduction. Note that in our case,  $\mathcal{A} = \{h_1, \dots, h_k\}' = \{f \in C^\infty(\Sigma) / \{h_j, f\} = 0, j = 1, \dots, k\}$ . Now we turn to describe Rieffel induction.

**Definition 2.1.** *Let  $\mathfrak{B}$  be a  $C^*$ -algebra. A Hilbert  $\mathfrak{B}$ -module is a  $\mathbb{C}$ -vector space right  $\mathfrak{B}$ -module  $X$  endowed with a pairing  $\langle \cdot, \cdot \rangle_{\mathfrak{B}} : X \times X \rightarrow \mathfrak{B}$  such that:*

1.  $\langle x, \lambda y + \mu z \rangle_{\mathfrak{B}} = \lambda \langle x, y \rangle_{\mathfrak{B}} + \mu \langle x, z \rangle_{\mathfrak{B}};$
2.  $\langle x, y \cdot b \rangle_{\mathfrak{B}} = \langle x, y \rangle_{\mathfrak{B}} b;$
3.  $\langle x, y \rangle_{\mathfrak{B}}^* = \langle y, x \rangle_{\mathfrak{B}}$
4.  $\langle x, x \rangle_{\mathfrak{B}} \geq 0$  (as an element of  $\mathfrak{B}$ );
5.  $\langle x, x \rangle_{\mathfrak{B}} = 0$  implies that  $x = 0$ ;
6.  $X$  is complete with the norm  $\|x\|_{\mathfrak{B}} := \|\langle x, x \rangle_{\mathfrak{B}}\|^{1/2}.$

If  $\overline{\text{span}\{x \cdot b : x \in X, b \in \mathfrak{B}\}} = X$  then  $X$  is called a non-degenerate Hilbert  $\mathfrak{B}$ -module.

In the literature sometimes some weaker conditions are asked, for instance conditions 5) and 6) are omitted and  $\mathfrak{B}$  can be replaced by a dense subalgebra  $\mathfrak{B}_0$  (in that case  $X$  is called a pre-Hilbert  $\mathfrak{B}_0$ -module). Nevertheless, after quotient by  $\mathcal{N}_0 := \{x \in X / \langle x, x \rangle_{\mathfrak{B}} = 0\}$ , take completion and extending the module product, we obtain a Hilbert  $\mathfrak{B}$ -module; in fact many examples can be obtained in this way.

An operator  $T : X \rightarrow X$  is called adjointable if there exist an operator  $T^* : X \rightarrow X$  such that  $\langle T(x), y \rangle_{\mathfrak{B}} = \langle x, T^*(y) \rangle_{\mathfrak{B}}$ . It turn out that every adjointable operator is a bounded linear  $\mathfrak{B}$ -module map, and the space of adjointable operator  $\mathcal{L}(X)$ , is a  $C^*$ -algebra with respect to the operator norm and its obvious involution.

Let  $\pi : \mathfrak{B} \rightarrow B(\mathcal{H}_\pi)$  be a representation. Then the algebraic tensor product  $X \odot \mathcal{H}_\pi$  can be endowed with a positive semi-definite inner product  $(\cdot, \cdot)$  such that  $(x \otimes u, y \otimes v) = (\pi(\langle x, y \rangle_{\mathfrak{B}})u, v)_{\mathcal{H}_\pi}$ . After quotient  $X \odot \mathcal{H}_\pi$  by the subspace of vectors with null length, we take completion and obtain a new Hilbert space  $\mathcal{H}^\pi$ . We denote by  $x \otimes_{\mathfrak{B}} u$  the image of  $x \odot u$  in  $\mathcal{H}^\pi$ ; then  $x \cdot b \otimes_{\mathfrak{B}} u = x \otimes_{\mathfrak{B}} \pi(b)u$ . Let  $\mathfrak{A}$  be a  $C^*$ -algebra and suppose that it acts on  $X$  by adjointable operators, in other words assume that we have a homomorphism of  $\mathfrak{A}$  into  $\mathcal{L}(X)$ . Then it can be shown that there is a

representation  $\text{Ind}\pi : \mathfrak{A} \rightarrow \mathcal{B}(\mathcal{H}^\pi)$  such that  $\text{Ind}\pi(a)(x \otimes_{\mathfrak{B}} u) = (a \cdot x \otimes_{\mathfrak{B}} u)$ . In particular we can induce representations of  $\mathcal{L}(X)$  from representations of  $\mathfrak{B}$ . The above described procedure is called Rieffel induction. The reader can find in [27] a comprehensive overview of Hilbert  $C^*$ -modules and Rieffel induction.

Summarizing, the quantum counterpart of the classical setting  $(\Sigma, P, J : \Sigma \rightarrow P^-)$  is a Hilbert  $\mathfrak{B}$ -module  $X$  and the quantum counterpart of Landsman-Xu generalization (the generalization of Marsden-Weinstein procedure) is Rieffel induction. So, the quantum counterpart of the construction of the symplectic manifolds  $\Sigma^\rho$  and the Poisson map  $\text{ind}\rho$  from  $(J^*[C^\infty(P)])'$  to  $C^\infty(\Sigma^\rho)$ , is the construction of the Hilbert space  $\mathcal{H}^\pi$  and the representation  $\text{Ind}\pi$  of  $\mathcal{L}(X)$  in  $\mathcal{B}(\mathcal{H}^\pi)$ .

## 2.2 Measurable fields of Hilbert spaces, Hilbert $C^*$ -modules and direct integrals.

In this subsection we start by summarizing some well known facts that we need concerning the theory of measurable (resp. continuous) field of Hilbert spaces MFHS (resp. CFHS) and direct integrals. We also study certain Hilbert  $C^*$ -modules which naturally arise from each MFHS (resp. CFHS). Applying Rieffel induction to those Hilbert  $C^*$ -modules, we proved that the corresponding algebra of adjointable operators is isomorphic to the algebra of decomposable operators; this is one of the main results of this article and together with the previous subsection lead to a justification to our quantization proposal. Finally, we showed some simple result that we will need later. For the theory of measurable field of Hilbert spaces and direct integrals we follow [6].

In what follows, either  $\Lambda$  denote a locally compact Hausdorff space or  $(\Lambda, \mathcal{B}, \eta)$  denote a measure space, i.e.  $\Lambda$  is a set,  $\mathcal{B}$  is a  $\sigma$ -algebra of subsets of  $\Lambda$  and  $\eta$  is a positive measure. We assume that  $(\Lambda, \mathcal{B}, \eta)$  is  $\sigma$ -finite and complete.

Assume that to each  $\lambda \in \Lambda$  we can associate a Hilbert space  $\mathcal{H}(\lambda)$  (this is usually called a field of Hilbert spaces over  $\Lambda$ ). We denote by  $\langle \cdot, \cdot \rangle_\lambda$  and  $\|\cdot\|_\lambda$  the inner product and norm in  $\mathcal{H}(\lambda)$ .

**Definition 2.2.** *We say that the  $\{\Lambda \ni \lambda \rightarrow \mathcal{H}(\lambda)\}$  is a measurable (resp. continuous) field of Hilbert spaces MFHS (resp. CFHS) if there is given a linear subspace  $\mathcal{S}$  of  $\prod_{\lambda \in \Lambda} \mathcal{H}(\lambda)$  possessing the following properties:*

- (i) *For every  $x \in \mathcal{S}$ , the function  $\lambda \rightarrow \|x(\lambda)\|_\lambda$  is  $\eta$ -measurable (resp. continuous);*
- (ii) *If  $y \in \prod_{\lambda \in \Lambda} \mathcal{H}(\lambda)$  is such that, for every  $x \in \mathcal{S}$ , the function  $\lambda \rightarrow \langle x(\lambda), y(\lambda) \rangle_\lambda$  is  $\eta$ -measurable (resp. the function  $\lambda \rightarrow \|x(\lambda) - y(\lambda)\|_\lambda$  is continuous), then  $y \in \mathcal{S}$ ;*
- (iii) *There exists a sequence  $(x_1, x_2, \dots)$  of elements of  $\mathcal{S}$  such that, for every  $\lambda \in \Lambda$ , the  $x_n(\lambda)$  form a total sequence in  $\mathcal{H}(\lambda)$ .*

*The elements of  $\mathcal{S}$  are called measurable (resp. continuous) vector fields. The set  $\{\lambda \in \Lambda : \mathcal{H}(\lambda) \neq 0\}$  is called the support of the field.*

If  $S_c$  defines a CFHS  $\{\Lambda \ni \lambda \rightarrow \mathcal{H}(\lambda)\}$  and  $(\Lambda, \mathcal{B}, \eta)$  is a measure space such that  $\mathcal{B}$  contains the Borel  $\sigma$ -algebra, then

$$S = \{x \in \prod_{\lambda \in \Lambda} \mathcal{H}(\lambda) : \Lambda \ni \lambda \rightarrow \langle x(\lambda), y(\lambda) \rangle \text{ is } \eta\text{-measurable } \forall y \in S_c\}$$

defines a MFHS; if this is the case,  $S$  is said to be deduced from  $S_c$ .

Note that polarization formula and condition (i) implies that, for every  $x, y \in \mathcal{S}$ , the function  $\lambda \rightarrow \langle x(\lambda), y(\lambda) \rangle_\lambda$  is  $\eta$ -measurable (resp. continuous). Then, condition (ii) implies that the product of a  $\eta$ -measurable (resp. continuous) vector field with a  $\eta$ -measurable (resp. continuous) complex-valued function is a  $\eta$ -measurable (resp. continuous) vector field.

If  $(\Lambda, \mathcal{B}, \eta)$  is a measure space, let  $L^\infty(\eta)$  be the von Neumann algebra of measurable essentially bounded complex-valued functions on  $\Lambda$ . If  $\Lambda$  is a locally compact Hausdorff space, let  $BC(\Lambda)$  be the  $C^*$ -algebra of bounded continuous complex functions on  $\Lambda$ , and  $C_0(\Lambda)$  the  $C^*$ -algebra of continuous complex functions on  $\Lambda$  vanishing at infinity. We haven't found stated elsewhere the following simple result (except for the case  $C_0(\Lambda)$ ).

**Proposition 2.1.** *Let  $\mathcal{S}$  define a MFHS (resp. CFHS) as before. Let*

$$\mathcal{S}^\infty := \{x \in \mathcal{S} : \lambda \rightarrow \|x(\lambda)\|_\lambda \in L^\infty(\Lambda, \eta)\}.$$

*Then pointwise multiplication makes  $\mathcal{S}^\infty$  a right  $L^\infty(\eta)$ -module. Moreover, the pairing defined by*

$$\langle x, y \rangle(\lambda) := \langle x(\lambda), y(\lambda) \rangle_\lambda,$$

*makes  $\mathcal{S}^\infty$  a non-degenerate Hilbert  $L^\infty(\eta)$ -module (up to quotient by the space of those  $x$ 's vanishing  $\eta$ -almost everywhere). Analogously, if  $\mathcal{S}$  define a CFHS and one replace  $L^\infty(\eta)$  for  $BC(\Lambda)$  or  $C_0(\Lambda)$ , then one obtain a Hilbert  $BC(\Lambda)$ -module  $\mathcal{BS}$  and a Hilbert  $C_0(\Lambda)$ -module  $\mathcal{S}_0$ .*

*Proof.* Up to the facts mentioned after the above definition, the proof is straightforward, anyway let us check at least completeness for the measurable setting. Note that  $\|x\| := (\|\langle x, x \rangle\|_\infty)^{\frac{1}{2}} = \text{ssup}_{\lambda \in \Lambda} \{\|x(\lambda)\|_\lambda\}$ .

Let  $(x_n)$  be a Cauchy sequence in  $\mathcal{S}^\infty$ . Then  $(x_n(\lambda))$  is a Cauchy sequence in  $\mathcal{H}(\lambda)$ , for every  $\lambda$  not belonging to a certain negligible set  $C$ ; hence it converge to an element in  $\mathcal{H}(\lambda)$  which we denote  $x(\lambda)$ . Put  $x(\lambda) = 0$  for  $\lambda \in C$ . For every  $y \in \mathcal{S}$ ,  $\langle x_n(\lambda), y(\lambda) \rangle_\lambda$  converge  $\langle x(\lambda), y(\lambda) \rangle_\lambda$  almost everywhere, then  $\lambda \rightarrow \langle x(\lambda), y(\lambda) \rangle_\lambda$  is measurable; therefore  $x$  belongs to  $\mathcal{S}$ . Since  $\|x_n(\lambda)\|$  converge to  $\|x(\lambda)\|$  a.e., clearly  $x \in \mathcal{S}^\infty$ . Given  $\epsilon > 0$ , let us fix  $N$  such that  $\|x_n(\lambda) - x_m(\lambda)\|_\lambda < \epsilon$ , a.e. for every  $n, m > N$ . Then  $\|x_n(\lambda) - x(\lambda)\|_\lambda \leq \epsilon$  almost everywhere for  $n > N$ , therefore  $(x_n)$  converge to  $x$ . Non-degeneracy follows from the existence of approximate units.

For the continuous setting the proof is practically the same (use first (ii) and uniform convergence in compact sets).  $\square$

It is well known that, when  $\Lambda$  is a Hausdorff locally compact space, this is the only way, up to isomorphism, to construct a Hilbert  $C_0(\Lambda)$ -module, i.e. all of them come from a continuous field of Hilbert spaces by defining the pairing as we did. Below we explained in more detail this result.

Now we will focus for a while in the measurable setting.

**Definition 2.3.** *Let  $\{\Lambda \ni \lambda \rightarrow \mathcal{H}(\lambda)\}$  be a measurable field of Hilbert spaces. Up to quotient by the space of measurable vector fields vanishing almost everywhere, the direct integral  $\int_\Lambda^\oplus \mathcal{H}(\lambda) d\eta(\lambda)$  is the space of square integrable measurable vector fields, i.e.*

$$\int_\Lambda^\oplus \mathcal{H}(\lambda) d\eta(\lambda) := \{x \in \mathcal{S} : \int_\Lambda \|x(\lambda)\|_\lambda^2 d\eta(\lambda) < \infty\}$$

It is well known (for instance, see part II, chapter I, proposition 5 in [6]) that  $\int_{\Lambda}^{\oplus} \mathcal{H}(\lambda) d\eta(\lambda)$  is a Hilbert space with the inner product

$$\langle x, y \rangle := \int_{\Lambda} \langle x(\lambda), y(\lambda) \rangle_{\lambda} d\eta(\lambda).$$

**Definition 2.4.** Let  $\{\Lambda \ni \lambda \rightarrow \mathcal{H}(\lambda)\}$  be a measurable field of Hilbert spaces. A field of operators  $\{\Lambda \ni \lambda \rightarrow A(\lambda) \in \mathcal{B}(\mathcal{H}(\lambda))\}$  is called measurable if for each  $\eta$ -measurable vector field  $x$ , the field  $\Lambda \ni \lambda \rightarrow A(\lambda)x(\lambda)$  is also  $\eta$ -measurable. Such field is called essentially bounded if the  $\eta$ -essential supremum of the map  $\lambda \rightarrow \|A(\lambda)\|$  is finite. Each essentially bounded measurable operator field define fiberwise an operator on  $\int_{\Lambda}^{\oplus} \mathcal{H}(\lambda) d\eta(\lambda)$ , such operator is called a decomposable operator.

Another well known fact ( see Theorem 1 in chapter 2, part II and its corollary in [6]) that we will need is the following:

**Proposition 2.2.** Let  $\{\Lambda \ni \lambda \rightarrow \mathcal{H}(\lambda)\}$  be a  $\eta$ -measurable field of Hilbert spaces. Then the algebra of decomposable operators is the commutant of the algebra of operators on  $\int_{\Lambda}^{\oplus} \mathcal{H}(\lambda) d\eta(\lambda)$  given by pointwise multiplication by elements of  $L^{\infty}(\eta)$  (such operators are called diagonalisable).

Lets denote by  $\mathfrak{C}^{\infty}$  the  $C^*$ -algebra of adjointable operators of  $\mathcal{S}^{\infty}$ . It is quite straightforward to check that, the space of elements in  $\mathcal{S}^{\infty}$  vanishing outside a set of finite measure is dense in  $\int_{\Lambda}^{\oplus} \mathcal{H}(\lambda) d\eta(\lambda)$ . Moreover, if  $A \in \mathfrak{C}^{\infty}$ , the inequality

$$\begin{aligned} \|Ax(\lambda)\|_{\lambda}^2 &= \langle x(\lambda), (A^*Ax)(\lambda) \rangle_{\lambda} \leq \|x(\lambda)\|_{\lambda} \|A^*Ax(\lambda)\|_{\lambda} \leq \|x(\lambda)\|_{\lambda}^{\frac{3}{2}} \sqrt{\|(A^*A)^2x(\lambda)\|_{\lambda}} \cdots \leq \\ &\leq \|x(\lambda)\|_{\lambda}^{\sum_{i=0}^n (\frac{1}{2})^i} \|(A^*A)^{2^n}x(\lambda)\|_{\lambda}^{\frac{1}{2}} \|x\|_{\infty} \end{aligned}$$

implies that  $\|Ax(\lambda)\|_{\lambda}^2 \leq \|x\|_{\infty} \|A\|^2 \|x(\lambda)\|_{\lambda}^2$  and this shows that

$$A \left( \mathcal{S}^{\infty} \cap \int_{\Lambda}^{\oplus} \mathcal{H}(\lambda) d\eta(\lambda) \right) \subseteq \mathcal{S}^{\infty} \cap \int_{\Lambda}^{\oplus} \mathcal{H}(\lambda) d\eta(\lambda).$$

Therefore, each adjointable operator define a densely defined operator in  $\int_{\Lambda}^{\oplus} \mathcal{H}(\lambda) d\eta(\lambda)$ . Clearly, every decomposable operator define an adjointable operator on  $\mathcal{S}^{\infty}$ . Moreover:

**Lemma 2.1.** If an adjointable operator can be extended to  $\int_{\Lambda}^{\oplus} \mathcal{H}(\lambda) d\eta(\lambda)$  then such extension is a decomposable operator.

*Proof.* Let  $A$  be an adjointable operator that extend to a bounded operator. Clearly, the adjoint of the extension of  $A$ , as a bounded operator in a Hilbert space, is an extension of  $A^*$  (initially defined in the Hilbert  $C^*$ -module setting). Let  $f \in L^{\infty}(\eta)$  and lets denote by  $M(f)$  the corresponding diagonalisable operator (pointwise multiplication). Then, for each  $x, y \in \mathcal{S}^{\infty} \cap \int_{\Lambda}^{\oplus} \mathcal{H}(\lambda) d\eta(\lambda)$ ,

$$\begin{aligned} \langle AM(f)x, y \rangle &= \int_{\Lambda} \langle (AM(f)x)(\lambda), y(\lambda) \rangle_{\lambda} d\eta(\lambda) = \\ &= \int_{\Lambda} \langle f(\lambda)x(\lambda), A^*y(\lambda) \rangle_{\lambda} d\eta(\lambda) = \int_{\Lambda} \langle f(\lambda)(Ax)(\lambda), y(\lambda) \rangle_{\lambda} d\eta(\lambda) = \\ &= \langle M(f)Ax, y \rangle. \end{aligned}$$

Therefore  $A \in [M(L^{\infty}(\eta))]'$ . □



The computations before the previous lemma seems to suggest that, there is no reason to think that any adjointable operator can be extended, but surprisingly we proved this is true. Our main tool to do it is Rieffel induction.

**Lemma 2.2.** *Let  $\pi : L^\infty(\eta) \rightarrow \mathcal{B}(L^2(\eta))$  be the multiplication representation. Then the corresponding Hilbert space constructed through Rieffel induction is isomorphic to  $\int_\Lambda^\oplus \mathcal{H}(\lambda) d\eta(\lambda)$ .*

*Proof.* Just note that the map  $U : \mathcal{S}^\infty \odot L^2(\eta) \rightarrow \int_\Lambda^\oplus \mathcal{H}(\lambda) d\eta(\lambda)$  given by

$$U \left( \sum_j x_j \otimes \phi_j \right) (\lambda) = \sum_j \phi_j(\lambda) x_j(\lambda),$$

where  $x_j \in \mathcal{S}^\infty$  and  $\phi_j \in L^2(\eta)$ , define the required unitary operator, because

$$\langle x \otimes_{L^\infty(\eta)} \phi, y \otimes_{L^\infty(\eta)} \psi \rangle = \langle \pi(\langle x, y \rangle) \phi, \psi \rangle = \int_\Lambda \langle \phi(\lambda) x(\lambda), \psi(\lambda) y(\lambda) \rangle_\lambda d\eta(\lambda)$$

and the image of  $U$  is dense in  $\int_\Lambda^\oplus \mathcal{H}(\lambda) d\eta(\lambda)$ .  $\square$

**Theorem 2.1.** *The representation  $\text{Ind}\pi$  of  $\mathfrak{C}^\infty$  induced by  $\pi$  through Rieffel induction, define an isomorphism of  $\mathfrak{C}^\infty$  and the algebra of decomposable operators. In particular, every adjointable operator can be extended to  $\int_\Lambda^\oplus \mathcal{H}(\lambda) d\eta(\lambda)$ .*

*Proof.* Since  $\pi$  is faithful and  $\mathcal{S}^\infty$  is non-degenerate,  $\text{Ind}\pi$  is also faithful. By definition, we have that  $\text{Ind}\pi(A)(\sum \phi_j x_j) = \sum \phi_j A x_j$ , where  $x_j \in \mathcal{S}^\infty$  and  $\phi_j \in L^2(\eta)$ . Since the space of finite sums of products of  $L^2(\eta)$  functions with elements of  $\mathcal{S}^\infty$  is dense in  $\int_\Lambda^\oplus \mathcal{H}(\lambda) d\eta(\lambda)$  and  $\text{Ind}\pi(A)$  is bounded, lemma (2.1) implies our result.  $\square$

This characterization of decomposable operators is quite surprising, not only because in principle it could allows us to show that certain operator is decomposable without necessarily checking that there is certain measurable family of operators, but also because it seems that one only would need to study such operator restricted to  $\mathcal{S}^\infty \cap \int_\Lambda^\oplus \mathcal{H}(\lambda) d\eta(\lambda)$ . On the other hand,  $\mathcal{S}^\infty$  is not necessarily contained in  $\int_\Lambda^\oplus \mathcal{H}(\lambda) d\eta(\lambda)$  and to check that certain operator on certain Hilbert  $C^*$ -module is adjointable is not straightforward at all as in the Hilbert space case (for instance, it isn't enough to check that it is bounded because there is not a version of Riesz representation theorem for Hilbert  $C^*$ -module).

Let assume that the MFHS comes from a continuous one and denote by  $\mathfrak{C}_0$  and  $\mathfrak{B}\mathfrak{C}$  the algebras of adjointable operators of  $\mathcal{S}_0$  and  $\mathcal{B}\mathcal{S}$  respectively. Then  $\mathfrak{C}_0 \subset \mathfrak{B}\mathfrak{C} \subset \mathfrak{C}^\infty$ .

**Proposition 2.3.** *Let  $\pi_\lambda : BC(\Lambda) \rightarrow \mathbb{C}$  be the evaluation at  $\lambda$  representation. Then the corresponding Hilbert space constructed through Rieffel induction is isomorphic to  $\mathcal{H}(\lambda)$ . Moreover  $\text{Ind}\pi_\lambda(A)(x(\lambda)) = (Ax)(\lambda)$ , for every  $A \in \mathfrak{B}\mathfrak{C}$  and  $x \in \mathcal{B}\mathcal{S}$ . The representation  $\text{Ind}\pi|_{BC(\Lambda)}$  coincide with the representation  $\int_\Lambda^\oplus \text{Ind}\pi_\lambda d\eta(\lambda)$  given by*

$$([\int_\Lambda^\oplus \text{Ind}\pi_\lambda(A) d\eta(\lambda)]x)(\lambda_0) = \text{Ind}\pi_{\lambda_0}(A)x(\lambda_0).$$

*The same holds true if we replace  $BC(\Lambda)$ ,  $\mathcal{B}\mathcal{S}$  and  $\mathfrak{B}\mathfrak{C}$  for  $C_0(\Lambda)$ ,  $\mathcal{S}_0$  and  $\mathfrak{C}_0$  respectively.*

In particular, this result explains why every Hilbert  $C_0(\Lambda)$ -module comes from a CFHS in the way described before.

Note that, in general decomposable operators could admit several decompositions (although they differ at most in a set of null  $\eta$ -measure); this result implies that those coming from  $\mathfrak{BC}$  or  $\mathfrak{C}_0$  has a particular one; this comes from the fact that the evaluation maps are not well defined in  $L^\infty(\eta)$ .

*Proof.* We naturally identify the algebraic tensor product  $\mathcal{BS} \odot \mathbb{C}$  with  $\mathcal{BS}$ . Under that identification, the positive semi-definite inner product on  $\mathcal{BS}$  defined by Rieffel induction is clearly

$$(x, y)_\lambda = \langle x(\lambda), y(\lambda) \rangle_\lambda .$$

Moreover, it is known that the map  $\mathcal{BS} \ni x \rightarrow x(\lambda) \in \mathcal{H}(\lambda)$  is onto, therefore it define the required unitary operator. The equality  $\text{Ind}\pi_\lambda(A)(x(\lambda)) = (Ax)(\lambda)$  follows from the general construction of an induced representation (see the previous subsection). For the last claim, just note that the equality holds for finite sums of products of  $L^2(\Lambda, \eta)$  functions with elements of  $\mathcal{BS}$ , and that the set of those finite sums is dense in  $\int_\Lambda^\oplus \mathcal{H}(\lambda) d\eta(\lambda)$ . The proof for the vanishing at infinity setting is the same.  $\square$

Let  $H_1, \dots, H_k$  be a finite family of pairwise strongly commuting selfadjoint operators on a separable Hilbert space  $\mathcal{H}$ . We denote by  $C^*(H_1, \dots, H_k)$  the  $C^*$ -algebra defined by the functional calculus of this family, i.e.  $C^*(H_1, \dots, H_k) := \{f(H_1, \dots, H_k) : f \in C_0(\text{sp}(H_1, \dots, H_k))\}$ , where  $\text{sp}(H_1, \dots, H_k)$  is the so called joint spectrum of the family (this is a closed subset of  $\mathbb{R}^k$ ). Also denote by  $W^*(H_1, \dots, H_k)$  the weak closure of  $C^*(H_1, \dots, H_k)$ . One of the key ingredients to understand our viewpoint is the following well known result:

**Theorem 2.2.** *Let  $H_1, \dots, H_k$  be a finite family of pairwise strongly commuting selfadjoint operators on a separable Hilbert space  $\mathcal{H}$ . Then, there exist a Radon measure  $\eta$  supported on  $\text{sp}(H_1, \dots, H_k)$ , a measurable field of Hilbert spaces  $\{\text{sp}(H_1, \dots, H_k) \ni \lambda \rightarrow \mathcal{H}(\lambda)\}$  and a unitary operator  $T : \mathcal{H} \rightarrow \int_{\text{sp}(H_1, \dots, H_k)}^\oplus \mathcal{H}(\lambda) d\eta(\lambda)$ , such that:*

1. *the Gelfand isomorphism from  $C^*(H_1, \dots, H_k)$  to  $C_0(\text{sp}(H_1, \dots, H_k))$  extend to an isomorphism from  $W^*(H_1, \dots, H_k)$  to  $L^\infty(\text{sp}(H_1, \dots, H_k), \eta)$ ,*
2.  *$\mathcal{H}(\lambda) \neq 0$   $\eta$ -almost everywhere and*
3.  *$[Tf(H_1, \dots, H_k)u](\lambda) = f(\lambda)Tu(\lambda)$   $\eta$ -almost everywhere,  $\forall f \in L^\infty(\text{sp}(H_1, \dots, H_k), \eta)$ .*

**Remark 2.1.** Lets denote by  $\mathfrak{C}^\infty(H_1, \dots, H_k)$  the corresponding algebra of decomposable operators (or the corresponding algebra of adjointable operators). Recall that, by definition, two selfadjoint operator  $H'$  and  $H$  commute if their corresponding spectral measures mutually commute. Therefore,  $H$  commute with  $H_1, \dots, H_k$  iff  $f(H) \in W^*(H_1, \dots, H_k)'$ , or equivalently, iff  $Tf(H)T^{-1}$  belongs to  $\mathfrak{C}^\infty(H_1, \dots, H_k)$ , for each bounded Borel function  $f$ .

In the next subsection we will need to know what can be said if we replace  $\eta$  by a measure  $\mu$  such that  $\eta \leq \mu$  (i.e.  $\eta$  is absolutely continuous respect to  $\mu$ ). For that, we need to understand part of the construction of the elements and the proof of the above cited theorem. The forthcoming result in this line will lead us to support our formal claim.

Lets sketch briefly the (canonical) proof of the above theorem: Proposition 4 iii) of part I, chapter 7 in [6] insure, for each  $C^*$ -algebra  $\mathfrak{A}$  weakly dense on an abelian von Neumann algebra of operators on  $\mathcal{H}$ , the existence of a type of measure called *basic measures* (see definition 1, part 1 chapter 7 in [6]) which is defined over the spectrum of  $\mathfrak{A}$ . Then proposition 1 of part I, chapter 7 in [6] state that if  $\eta$  is a basic measure over the spectrum of an abelian  $C^*$ -algebra of operators, then Gelfand isomorphism can be extended to an isomorphism between the corresponding  $L^\infty$ -space and the weak closure of the  $C^*$ -algebra. The required measure and the first claim follow by applying those results to  $C^*(H_1, \dots, H_k)$  and  $W^*(H_1, \dots, H_k)$ .

Lets denote by  $\mathcal{G} : \mathfrak{A} \rightarrow C_0(\Lambda)$  the Gelfand isomorphism. Since  $\eta$  is basic, for each  $u, v \in \mathcal{H}$  the measure  $\eta_{u,v}$  given by the positive functional  $C_c(\Lambda) \ni f \rightarrow \langle \mathcal{G}^{-1}(f)u, v \rangle$  is absolutely continuous respect to  $\eta$ ; let  $h_{u,v}$  the corresponding Radon-Nikodym derivate. Let  $\mathcal{K}$  be the space algebraically generated by a fix a sequence  $(u_j)$  dense in  $\mathcal{H}$ . Then over  $\mathcal{K}$  the expression  $(u, v)_\lambda = h_{u,v}(\lambda)$  define a positive semi-definite inner product  $\eta$ -almost everywhere and  $\mathcal{H}(\lambda)$  is defined by completing the quotient of  $\mathcal{K}$  by the corresponding null length vector subspace. In the proof of theorem 1 of part II, chapter 6 in [6], it was shown that  $\eta_{u,v}(\{\lambda : \mathcal{H}(\lambda) = 0\}) = 0$  and then the same holds for  $\eta$ , since it is basic. Denote by  $u(\lambda)$  the image of  $u \in \mathcal{K}$  in  $\mathcal{H}(\lambda)$ . Since  $\langle u_j(\lambda), u_k(\lambda) \rangle_\lambda = (u_j, u_k)_\lambda = h_{u_j, u_k}(\lambda)$  is  $\eta$ -measurable and the sequence  $(u_j(\lambda))$  is total in  $\mathcal{H}(\lambda)$ , proposition 4 of part II chapter 1 in [6] implies the existence of the required  $\eta$ -measurable field of Hilbert spaces structure. The unitary operator is initially defined by  $T(\sum_m \mathcal{G}^{-1}(f_j)u_j)(\lambda) = \sum_j f_j(\lambda)u_j(\lambda)$ , where  $f_j \in L^\infty(\Lambda, \eta)$ . Since  $\mathcal{H}(\lambda) \neq 0$   $\eta$ -almost everywhere,  $T$  is well defined. Then it is extended by density and the rest of the properties follow.

Let  $\mu$  be a Radon measure and  $\mu \geq \eta$  over the Borel sigma algebra, assume that  $\Lambda$  is  $\sigma$ -compact and let  $N = \{\lambda \in \Lambda : \frac{d\eta}{d\mu}(\lambda) \neq 0\}$ . Note that  $N$  is Borel and  $\eta(N^c) = 0$ . The equality  $\eta(B) = \int_B \frac{d\eta}{d\mu}(\lambda) d\mu(\lambda)$  for  $B$  Borel together with the inner and outer regularity of  $\eta$  and  $\mu$ , implies that  $A \subset \Lambda$  is  $\eta$ -measurable iff  $A \cap N$  is  $\mu$ -measurable; also  $\eta(A) = 0$  iff  $\mu(A \cap N) = 0$ . Then a function on  $\Lambda$  is  $\eta$ -measurable iff  $f|_N$  is  $\mu$ -measurable. In particular, the restriction map  $r : L^\infty(\Lambda, \eta) \hookrightarrow L^\infty(N, \mu)$  is an isomorphism.

**Proposition 2.4.** *Let  $\mathcal{H}$  be a separable Hilbert space,  $\mathfrak{A}$  be an abelian  $C^*$ -algebra of operators on  $\mathcal{H}$  and  $\eta$  a basic measure for  $\mathfrak{A}$ . Assume that the identity belongs to the weak closure of  $\mathfrak{A}$  and that the spectrum of  $\mathfrak{A}$  is  $\sigma$ -compact. Let  $\mu \geq \eta$  a Radon measure and  $N, r$  as above. Then there exist a  $\mu$ -measurable field of Hilbert spaces  $\{N \ni \lambda \rightarrow \mathcal{H}^\mu(\lambda)\}$  and an unitary operator  $T^\mu$  of  $\mathcal{H}$  into  $\int_N^\oplus \mathcal{H}^\mu(\lambda) d\mu(\lambda)$ , which transform the extension of  $[r \circ \mathcal{G}]^{-1}$  into the canonical isomorphism between  $L^\infty(N, \mu)$  and the corresponding algebra of diagonalisable operators.  $\mathcal{H}^\mu(\lambda)$  coincide with  $\mathcal{H}(\lambda)$  as vector space and its inner product is  $\frac{d\eta}{d\mu}(\lambda)$  times the one of  $\mathcal{H}(\lambda)$ .*

In other words, we get almost the same result for  $\mu$  than for  $\eta$ , but the support of the corresponding field might be a proper subset of the spectrum of  $\mathfrak{A}$ .

*Proof.* Note that  $\eta_{u,v} \leq \eta \leq \mu$  and then  $d\eta_{u,v}/d\mu = h_{u,v} \cdot d\eta/d\mu$ . This implies that, if we follow the construction of the  $\eta$ -measurable field of Hilbert spaces  $\{\lambda \rightarrow \mathcal{H}(\lambda)\}$  described above but replacing  $\eta$  by  $\mu$ , we still get a  $\mu$ -measurable field of Hilbert spaces  $\{\text{sp}(\mathfrak{A}) \ni \lambda \rightarrow \mathcal{H}^\mu(\lambda)\}$ , such that as vector space

$$\mathcal{H}^\mu(\lambda) = \begin{cases} \mathcal{H}(\lambda) & \text{if } \lambda \in N \\ 0 & \text{otherwise.} \end{cases}$$

In the first case, the inner product in  $\mathcal{H}^\mu(\lambda)$  is  $\frac{d\eta}{d\mu}(\lambda)$  times the inner product of  $\mathcal{H}(\lambda)$ . After removing  $N^c$  from the field, the reminder of the proof follows in the same way than for  $\eta$ .  $\square$

### 2.3 Quantizing $\Sigma_\lambda$ and Landsman's approach.

Recall that our first goal is to quantize the symplectic manifolds  $\Sigma_\lambda$ . Assume that the symplectic manifold  $\Sigma$  has been quantized into a separable Hilbert space  $\mathcal{H}$ , i.e. there is a procedure that send suitable elements of  $C^\infty(\Sigma)$  into operators on  $\mathcal{H}$ ; in particular assume that the family  $h_1, \dots, h_k$  has been quantized to the family of selfadjoint operators  $H_1, \dots, H_k$ . Assume also that  $H_1, \dots, H_k$  strongly commute pairwise. Recall that  $\Sigma_\lambda$  comes from Marsden-Weinstein reduction or from the representation of the Poisson algebra  $\{h_1, \dots, h_k\}' := \{f \in C^\infty(\Sigma) : \{f, h_j\} = 0 \forall 1 \leq j \leq k\}$  induced by the inclusion Poisson map  $\{\lambda\} \hookrightarrow J(\Sigma)$ . We propose to quantize  $\Sigma_\lambda$  in a Hilbert space constructed following the same steps but quantum like, i.e. using the quantum counterpart of Marsden-Weinstein reduction: according to subsection (2.1), we should apply Rieffel induction over certain Hilbert  $C^*$ -module having  $\mathfrak{C}^\infty(H_1, \dots, H_k)$ , the algebra of operators that commute with  $H_1, \dots, H_k$ , as an algebra of adjointable operators, so we would get a representation of this algebra on the Hilbert space we are looking for. According to [28], the ambient trivial Poisson algebra  $C^\infty(\mathbb{R}^k)$  should be quantized into the group  $C^*$ -algebra  $C^*(\mathbb{R}^k) \cong C_0(\mathbb{R}^k)$ , this suggest that we should look for a  $C_0(\text{sp}(H_1, \dots, H_k))$ -module. Assume that the MFHS associated with  $H_1, \dots, H_k$  given by theorem (2.2) comes from a continuous one. Our main heuristic claim is that the natural Hilbert  $C^*$ -module to be considered as the required quantum counterpart of our classical setting is the  $C_0(\text{sp}(H_1, \dots, H_k))$ -module  $\mathcal{S}_0(H_1, \dots, H_k)$ . We also claim that the quantum counterpart of the classical Poisson map  $\{\lambda\} \hookrightarrow J(\Sigma)$  is the representation  $\pi_\lambda$  of  $C_0(\text{sp}(H_1, \dots, H_k))$ . Therefore, according to proposition (2.3),  $\Sigma_\lambda$  should be quantized into  $\mathcal{H}(\lambda)$ .

Of course, this is just a formal statement, perhaps well justified in this section, but actually it seems to fail in general (at least in this current form). Below we will give an example. The obvious problem is that in general a regular  $\lambda \in J(\Sigma)$  doesn't necessarily belongs to  $\text{sp}(H_1, \dots, H_k)$  (not even  $\eta$ -almost everywhere). At the end of this subsection we will propose certain condition that we believe it could be important to deal with this issue. We should note that at least for the  $\hbar$ -dependent quantization procedures we have in mind, which by simplicity here we denote by  $\mathfrak{Op}^\hbar$  (this is the standard notation for the Weyl calculus, for its magnetic generalization [20] one should add also dependence on a vector potential), this problem in the semiclassical limit seems to vanish in some sense. More precisely for  $k = 1$ , results of the kind

$$\lim_{\hbar \rightarrow 0} \text{sp}(\mathfrak{Op}^\hbar(h)) = \overline{h(\mathbb{R}^{2n})},$$

in a suitable sense and for certain Hamiltonians, can be found at least in [1] [4] (also for any  $k$ , results on this line but considering the convex hulls of the involved sets are in [26]). So, formally one could expect that, once  $\lambda \in h(\mathbb{R}^{2n})$  is given, one can find  $\hbar$  small enough such that  $\lambda \in \text{sp}(\mathfrak{Op}^\hbar(h))$  too. However, any procedure that deserve to be called "a quantization of  $\Sigma_\lambda$ " should give sense to many suitable semiclassical problems, so at least we need Hamiltonians satisfying that there is  $\hbar_0$  small enough such that  $h(\mathbb{R}^{2n}) \subseteq \text{sp}(\mathfrak{Op}^\hbar(h))$  for every  $\hbar < \hbar_0$ .

On the other hand, in the forthcoming sections we will develop cases where  $\text{sp}(H_1, \dots, H_k) = \overline{J(\Sigma)}$ ,  $\eta = m$  is the Lebesgue measure and our formal claim will seem clear enough, so this will not be a problem.

We want to describe a formal approach to this statement due to Landsman (it also follows from the global idea that Rieffel induction is the quantum counterpart of the generalized Marsden-Weinstein reduction), which was the starting point to develop ours, and we also want to explain the relationship between both approaches. Let us "quantize" the elements used to construct  $\Sigma_\lambda$ . It is well known that the flow  $\Phi_x$  should be quantized to the one parameter group  $U(x) := e^{ix \cdot H}$ ,

where  $x \cdot H := \sum_j^k x_j H_j$ . Also, according to [28], the trivial Poisson algebra  $C^\infty(\mathbb{R}^k)$  should be quantize into the group  $C^*$ -algebra  $C^*(\mathbb{R}^k)$ . So the plan again consist in defining naturally a Hilbert  $C^*(\mathbb{R}^k)$ -module and applying Rieffel induction once we choose certain representations of  $C^*(\mathbb{R}^k)$ . The integral form of  $U$  is the representation  $r : C^*(\mathbb{R}^k) \rightarrow B(\mathcal{H})$ , given by

$$r(f) = \int_{\mathbb{R}^k} f(x)U(-x)dx,$$

initially defined, for instance, in  $C_c(\mathbb{R}^k)$  and then extended. This automatically make  $\mathcal{H}$  a module over  $C^*(\mathbb{R}^k)$ :  $f \cdot u = r(f)u$ .

Landsman suggested to endow a suitable dense subspace of  $\mathcal{H}$  with a Hilbert  $C^*(\mathbb{R}^k)$ -module structure; in fact, he proposed to look for a subspace  $\hat{X}$  stable under  $r(f)$  (for each  $f \in C_c^\infty(\mathbb{R}^k)$ ) and for which the function  $x \rightarrow (u, U(x)v)_{\mathcal{H}}$  lies in  $C_c^\infty(\mathbb{R}^k)$  for all  $u, v \in \hat{X}$ .

He also proved that for any amenable Lie group  $G$  (one must replace above  $\mathbb{R}^k$  by  $G$  and the unitary representation  $U$  for any representation of  $G$ ) and a dense subspace  $\hat{X}$  satisfying the above conditions,  $\hat{X}$  becomes a pre-Hilbert  $C_c^\infty(G)$ -module with the pairing  $\langle u, v \rangle_{C_c^\infty(G)} = (u, U(\cdot)v)_{\mathcal{H}}$  (theorem 2.5.4 in [16]).

It is not clear to us that such space  $\hat{X}$  exists in general. Below, we are going to assume that  $\eta$  is absolutely continuous respect to the Lebesgue measure, this will lead us to some constraints to this problem and to relate this group-like approach to ours.

First recall that the Fourier transform  $\mathcal{F}$  on  $C_c^\infty(\mathbb{R}^k)$  extend to an isomorphism (which we also denote by  $\mathcal{F}$ ) of  $C^*(\mathbb{R}^k)$  onto  $C_0(\mathbb{R}^k)$ . Then if the subspace  $\hat{X}$  exists, it also gives a Hilbert  $C_0(\mathbb{R}^k)$ -module, just by defining  $\phi \cdot u = r \circ \mathcal{F}^{-1}(\phi)u$  and  $\langle u, v \rangle_{C_0(\mathbb{R}^k)} = \mathcal{F}(\langle u, v \rangle_{C^*(\mathbb{R}^k)})$ . In particular,  $\hat{X}$  would be isomorphic (as a pre-Hilbert module) to the  $\mathcal{S}_0$  space of continuous sections vanishing at infinity of certain CFHS. Let us calculate  $r \circ \mathcal{F}^{-1}$ .

$$\begin{aligned} \langle r(f)u, v \rangle &= \langle Tr(f)u, Tv \rangle = \int_{\mathbb{R}^k} \left\langle \left( \int_{\mathbb{R}^k} f(x)TU(-x)u dx \right) (\lambda), Tv(\lambda) \right\rangle_{\mathcal{H}(\lambda)} d\eta(\lambda) \\ &= \int_{\mathbb{R}^k} \left\langle \left( \int_{\mathbb{R}^k} f(x)e^{-ix \cdot \lambda} dx \right) Tu(\lambda), Tv(\lambda) \right\rangle_{\mathcal{H}(\lambda)} d\eta(\lambda) = \\ &= \int_{\mathbb{R}^k} \langle (\mathcal{F}f)(\lambda)Tu(\lambda), Tv(\lambda) \rangle_{\mathcal{H}(\lambda)} d\eta(\lambda) = ((\mathcal{F}f)(H_1, \dots, H_k)u, v). \end{aligned}$$

So  $r \circ \mathcal{F}^{-1}$  is just the standard functional calculus of  $H_1, \dots, H_k$ .

The required field of Hilbert spaces can be obtained applying Rieffel induction over the evaluation representations  $\pi_\lambda$ . Since  $\hat{X} \odot \mathbb{C} \cong \hat{X}$ , in order to identify those Hilbert spaces we need to calculate  $\mathcal{F} \circ \langle u, v \rangle_{C_c^\infty(\mathbb{R}^k)}$ . In what follows assume that  $\eta \leq m$ , i.e.  $\eta$  is absolutely continuous respect to the Lebesgue measure, then we have

$$\begin{aligned} (\mathcal{F} \langle u, v \rangle_{C_c^\infty(\mathbb{R}^k)})(\xi) &= \int_{\mathbb{R}^k} \langle u, U(x)v \rangle e^{-ix \cdot \xi} dx = \int_{\mathbb{R}^k} \left[ \int_{\mathbb{R}^k} \langle Tu(\lambda), e^{ix \cdot \lambda} Tv(\lambda) \rangle d\eta(\lambda) \right] e^{-ix \cdot \xi} dx = \\ &= \int_{\mathbb{R}^k} \int_{\mathbb{R}^k} \langle Tu(\lambda), Tv(\lambda) \rangle \frac{d\eta}{dm}(\lambda) e^{ix \cdot (\lambda - \xi)} d\lambda dx = \langle Tu(\xi), Tv(\xi) \rangle \frac{d\eta}{dm}(\xi), \end{aligned}$$

where  $\frac{d\eta}{dm}$  is the Radon-Nikodym derivative of the measure  $\eta$  respect to the Lebesgue measure. In other words, the required pre-Hilbert  $C_c(\mathbb{R}^k)$ -module would be isomorphic to a pre-Hilbert

$\mathcal{F}(C_c(\mathbb{R}^k))$ -module of vanishing at infinity continuous sections for the MFHS given by proposition (2.4), as in the previous subsection. Conversely, if the MFHS given by Theorem (2.2) is deduced from a continuous one and the corresponding measure  $\eta$  is absolutely continuous respect to  $m$ , using inverse Fourier transform and  $T$ , we can define the required subspace  $\hat{X}$ . In particular, both approaches suggest that  $\Sigma_\lambda$  should be quantized by  $\mathcal{H}(\lambda)$ .

The last computation was done for  $\xi = 0$  in [15] (assuming that  $0 \in \text{sp}(H_1)$ ) with the same purpose than us, i.e. looking for a proper Hilbert space to quantize  $\Sigma_0$ , but in the language of constrained systems and comparing it with P. Dirac's proposal. It was also found following Rieffel induction presented in a different but equivalent manner.

Finally lets show a counterexample. Let  $\Sigma = \mathbb{R}^{2n}$ , endowed with the canonical symplectic structure, and let  $h(x, \xi) = 1/2(|x|^2 + |\xi|^2)$  the harmonic oscillator Hamiltonian. Then  $\Sigma_\lambda \cong CP^{n-1}$ , for every  $\lambda > 0$  (see example 4.3.4(iv) in [2] for details). Using the Weyl quantization, we get the quantum harmonic oscillator operator  $H = 1/2(-\Delta + |x|^2)$  on  $L^2(\mathbb{R}^n)$ . Then  $\text{sp}(H) = \mathbb{N} + n/2$  is discrete. So clearly, one can choose  $\eta$  to be the counting measure and  $\mathcal{H}(\lambda \equiv j + n/2)$  to be the corresponding eigenspace. Then of course, our heuristic claim cannot be follow in its current form (even if we introduce Planck's constant and make it "small enough"). This example and Landsman's approach seem to suggest that at least we should restrict our attention to absolutely continuous Hamiltonians.

### 3 The main explicit cases

In this section we give the type of Hamiltonians we are going to consider in the reminder of the article. In this cases we have that  $\text{sp}(H_1, \dots, H_k) = \overline{J(\mathbb{R}^{2n})}$  and we are going to compute explicitly both  $\Sigma_\lambda$  and  $\mathcal{H}(\lambda)$  for each regular  $\lambda \in J(\mathbb{R}^{2n})$ . For the moment, let  $h_j(x, \xi) = \phi_j(x)$ , where  $\phi_j \in C^\infty(\mathbb{R}^n)$ . Also let, for each  $\lambda = (\lambda_1, \dots, \lambda_k) \in J(\mathbb{R}^{2n})$ ,

$$\tilde{\Sigma}_\lambda := \bigcap_j \phi_j^{-1}(\lambda_j)$$

and

$$\Phi_{t_1, \dots, t_k} := \Phi_{t_1}^1 \circ \dots \circ \Phi_{t_k}^k.$$

Then

$$\hat{\Sigma}_\lambda := J^{-1}(\lambda) = \tilde{\Sigma}_\lambda \times \mathbb{R}^n$$

and

$$\Phi_{t_1, \dots, t_k}(x, \xi) = (x, \xi + t_1 \nabla \phi_1(x) + \dots + t_k \nabla \phi_k(x)) \quad \forall t_j \in \mathbb{R}, (x, \xi) \in \mathbb{R}^{2n}.$$

If we assume that the range of the matrix  $(\partial_l \nabla \phi_m(x))$  is  $k$  for every  $x \in \tilde{\Sigma}_\lambda$  then, by the implicit function theorem,  $\tilde{\Sigma}_\lambda$  is a  $n - k$  submanifold of  $\mathbb{R}^n$  and  $\nabla \phi_j(x)$  is normal at each  $x \in \tilde{\Sigma}_\lambda$ , i.e.  $\nabla \phi_j(x) \in [i_*^\lambda(T_x \tilde{\Sigma}_\lambda)]^\perp$ , where  $i^\lambda : \tilde{\Sigma}_\lambda \rightarrow \mathbb{R}^n$  is the obvious inclusion and  $T_x \mathbb{R}^n$  is identified with  $\mathbb{R}^n$  in the usual way.

Let  $g$  be the metric on  $\tilde{\Sigma}_\lambda$  induced from  $\mathbb{R}^n$ , i.e.

$$g_x : T_x \tilde{\Sigma}_\lambda \times T_x \tilde{\Sigma}_\lambda \rightarrow \mathbb{R} \tag{1}$$

$$g_x(v, w) := \langle i_*^\lambda(v), i_*^\lambda(w) \rangle.$$

Also, let  $\tilde{g}_x : T_x \tilde{\Sigma}_\lambda \rightarrow T_x^* \tilde{\Sigma}_\lambda$  be the natural isomorphism coming from  $g_x$ . The map  $i_*^\lambda \circ \tilde{g}_x^{-1}$  allow us to identify  $T_x^* \tilde{\Sigma}_\lambda$  with the linear subspace  $\langle \nabla \phi_1(x), \dots, \nabla \phi_k(x) \rangle^\perp$ . Lets denote by  $q_x : \mathbb{R}^n \rightarrow \langle \nabla \phi_1(x), \dots, \nabla \phi_k(x) \rangle^\perp$  the projection on that space.

Perhaps the following result is clear from the point of view of a “symplectic geometer”, but i didn’t find it elsewhere, not even stated or as an exercise, neither i could put it as a corollary of some general result, and it is very important for our purposes.

**Theorem 3.1.** *Let  $\phi_j \in C^\infty(\mathbb{R}^n)$  and  $h_j(x, \xi) = \phi_j(x)$ . Assume that the range of the matrix  $(\partial_l \phi_m(x))$  is  $k$  for every  $x \in \tilde{\Sigma}_\lambda$ , then:*

$$\Sigma_\lambda \ni [(x, \xi)] \rightarrow (x, q_x(\xi)) \in T^* \tilde{\Sigma}_\lambda \quad (2)$$

is a symplectomorphism, where  $[(x, \xi)]$  denote the orbit of  $(x, \xi)$  by  $\Phi$ . Here  $T^* \tilde{\Sigma}_\lambda$  is endowed with the standard symplectic structure on a cotangent bundle and  $T_x^*(\tilde{\Sigma}_\lambda)$  is identified as above with the  $(n - k)$ -dimensional subspace  $\langle \nabla \phi_1(x), \dots, \nabla \phi_k(x) \rangle^\perp$ .

We decided to state this theorem in this way to emphasize its geometrical meaning, otherwise the reader should replace “ $q_x(\xi)$ ” by “ $\tilde{g}_x \circ (i_*^\lambda)^{-1}(q_x(\xi))$ ”.

*Proof.* It is clear that the map (2) is well defined and a diffeomorphism. The closed 2-form  $\Omega^\lambda$  in  $\Sigma_\lambda$  is determined by the equation

$$(i^\lambda)^* \Omega = (\pi^\lambda)^* \Omega^\lambda,$$

where  $\Omega = \sum_k d\xi_k \wedge dx_k$  is the standard closed 2-form on  $\mathbb{R}^{2n}$  and  $\pi^\lambda : \hat{\Sigma}_\lambda \rightarrow \Sigma_\lambda$  is the quotient map onto the orbit space (see [23] or [19] or [18]). So we only need to check that the standard closed 2-form in  $T^*(\tilde{\Sigma}_\lambda)$  satisfies that equation after replacing  $\pi^\lambda$  by  $\tilde{\pi}^\lambda : \hat{\Sigma}_\lambda \rightarrow T^*(\tilde{\Sigma}_\lambda)$  given by  $\tilde{\pi}^\lambda((x, \xi)) = (x, q_x(\xi))$ . During this proof, we will denote this form by  $\tilde{\Omega}^\lambda$ .

Fix  $(x_0, \xi_0) \in \hat{\Sigma}_\lambda$ , it is enough to prove that there is a basis  $\{v_j\}_{j=1}^{2n-k}$  for  $T_{(x_0, \xi_0)} \hat{\Sigma}_\lambda$  such that

$$\Omega_{(x_0, \xi_0)}(i_*^\lambda(v_l), i_*^\lambda(v_m)) = \tilde{\Omega}_{\tilde{\pi}^\lambda((x_0, \xi_0))}^\lambda(\tilde{\pi}_*^\lambda(v_l), \tilde{\pi}_*^\lambda(v_m)) \quad (3)$$

Let us fix a chart  $(z_1, \dots, z_{n-k}; U)$  at  $x_0 \in \tilde{\Sigma}_\lambda$  and let  $\{\frac{\partial}{\partial z_j}|_{x_0}\}_{j=1}^{n-k}$  the corresponding base of  $T_x \tilde{\Sigma}_\lambda$ . Since  $\hat{\Sigma}_\lambda$  is diffeomorphic to  $\tilde{\Sigma}_\lambda \times \mathbb{R}^n$ , we can consider

$$v_j = \begin{cases} \frac{\partial}{\partial z_j}|_{(x_0, \xi_0)} & 1 \leq j \leq n - k \\ \frac{\partial}{\partial \xi_{(j+k-n)}}|_{(x_0, \xi_0)} & n - k < j \leq 2n - k. \end{cases}$$

Let us calculate the left hand side of (3). Let  $w_j = i_*^\lambda(\frac{\partial}{\partial z_j}|_{x_0})$ , then obviously

$$[i^\lambda]_*(v_j) = \begin{cases} \sum_m \langle w_j, e_m \rangle \frac{\partial}{\partial x_m}|_{(x_0, \xi_0)} & j \leq n - k \\ \frac{\partial}{\partial \xi_{j+k-n}}|_{(x_0, \xi_0)} & j > n - k \end{cases}$$

By definition

$$\Omega_{(x_0, \xi_0)}(i_*^\lambda(v_j), i_*^\lambda(v_l)) = \frac{1}{2} \sum_m \det \begin{pmatrix} d\xi_m|_{(x_0, \xi_0)}(i_*^\lambda(v_j)) & d\xi_m|_{(x_0, \xi_0)}(i_*^\lambda(v_l)) \\ dx_m|_{(x_0, \xi_0)}(i_*^\lambda(v_j)) & dx_m|_{(x_0, \xi_0)}(i_*^\lambda(v_l)) \end{pmatrix}.$$

So, for instance, if  $j > n - k$  and  $l \leq n - k$  we get

$$\frac{1}{2} \sum_m \det \begin{pmatrix} d\xi_m|_{(x_0, \xi_0)} \left( \frac{\partial}{\partial \xi_{j+k-n}}|_{(x_0, \xi_0)} \right) & d\xi_m|_{(x_0, \xi_0)} \left( \sum_{r=1}^n < w_l, e_r > \frac{\partial}{\partial x_r}|_{(x_0, \xi_0)} \right) \\ dx_m|_{(x_0, \xi_0)} \left( \frac{\partial}{\partial \xi_{j+k-n}}|_{(x_0, \xi_0)} \right) & dx_m|_{(x_0, \xi_0)} \left( \sum_{r=1}^n < w_l, e_r > \frac{\partial}{\partial x_r}|_{(x_0, \xi_0)} \right) \end{pmatrix} =$$

$$= 1/2 < w_l, e_{j+k-n} > .$$

The other cases can be obtained in the same way and we get

$$\Omega_{(x_0, \xi_0)}(i_*^\lambda(v_j), i_*^\lambda(v_l)) = \begin{cases} -1/2 < w_j, e_{l+k-n} > & j \leq n - k, l > n - k \\ 1/2 < w_l, e_{j+k-n} > & j > n - k, l \leq n - k \\ 0 & \text{otherwise.} \end{cases}$$

For the right hand side, let us denote by  $(z, p)$  the elements of  $T^*\tilde{\Sigma}_\lambda$  and by  $p_j$  the coordinates on the cotangent part corresponding to the dual base  $dz_j|_z$ . By definition,

$$\tilde{\pi}_*^\lambda(v_j) = \sum_{m=1}^{n-k} a_m \frac{\partial}{\partial z_m}|_{\tilde{\pi}^\lambda((x_0, \xi_0))} + b_m \frac{\partial}{\partial p_k}|_{\tilde{\pi}^\lambda((x_0, \xi_0))}$$

where  $a_m = v_j(z_m \circ \tilde{\pi}^\lambda)$  and  $b_m = v_j(p_m \circ \tilde{\pi}^\lambda)$ . It is clear that  $z_m \circ \tilde{\pi}^\lambda(x, \xi) = z_m(x)$ .

Let  $y_l := i_*^\lambda \circ \tilde{g}_{x_0}^{-1}(dz_l|_{x_0})$ . Clearly  $\{y_l\}_{l=1}^{n-k}$  is a base of  $i_*^\lambda(T_{x_0}\tilde{\Sigma}_\lambda)$ .

Note that

$$< y_l, w_m > = g_{x_0}(\tilde{g}_{x_0}^{-1}(dz_l|_{x_0}), \frac{\partial}{\partial z_m}|_{x_0}) = dz_l|_{x_0}(\frac{\partial}{\partial z_m}) = \delta_l^m.$$

Then for every  $\xi \in \mathbb{R}^n$

$$q_{x_0}(\xi) = \sum_l < \xi, w_l > y_l.$$

Therefore

$$p_m \circ \tilde{\pi}^\lambda(x, \xi) = p_m \left[ \tilde{g}_x \circ (i_*^\lambda)^{-1} \left( \sum_{l=1}^{n-k} < \xi, w_l > y_l \right) \right] = p_m \left( \sum_{l=1}^{n-k} < \xi, w_l > dz_l \right) = < \xi, w_m > .$$

So, for  $j > n - k$ ,  $v_j(p_m \circ \tilde{\pi}^\lambda) = < e_{j+k-n}, w_m > .$  Then

$$[\pi_\lambda]_*(v_j) = \begin{cases} \frac{\partial}{\partial z_j}|_{\tilde{\pi}^\lambda((x_0, \xi_0))} & j \leq n - k \\ \sum_m < e_{j+k-n}, w_m > \frac{\partial}{\partial p_m}|_{\tilde{\pi}^\lambda((x_0, \xi_0))} & j > n - k. \end{cases}$$

Finally, computing in the same way that for the left hand side, we get

$$\hat{\Omega}_{\tilde{\pi}^\lambda((x_0, \xi_0))}(\tilde{\pi}_*^\lambda(v_j), \tilde{\pi}_*^\lambda(v_l)) = \begin{cases} -1/2 < w_j, e_{l+k-n} > & j \leq n - k, l > n - k \\ 1/2 < w_l, e_{j+k-n} > & j > n - k, l \leq n - k \\ 0 & \text{otherwise.} \end{cases}$$

□



Let us pass to the quantum side. In this context, we quantize classical Hamiltonians in  $\mathbb{R}^{2n}$  into selfadjoint operators over  $L^2(\mathbb{R}^n)$  through the Weyl quantization (calculus)  $\mathfrak{Op}$ . During this section, we will not need to introduce  $\hbar$ -dependence of  $\mathfrak{Op}$ .

If  $h_j(x, \xi) = \phi_j(x)$  as before, then  $H_j := \mathfrak{Op}(h_j) = \phi_j(Q)$  is the operator multiplication by  $\phi_j$ . Clearly,  $\{H_j\}_{j=1}^k$  is pairwise commuting family of selfadjoint operators. Let  $\tilde{J} : \mathbb{R}^n \rightarrow \mathbb{R}^k$  be given by  $\tilde{J}(x) = (\phi_1(x), \dots, \phi_k(x))$ . Then the functional calculus is given by  $f(H_1, \dots, H_k)u(x) = [f \circ \tilde{J}](x)u(x)$ , where  $f$  is a bounded Borel function on  $\mathbb{R}^k$  and  $u \in L^2(\mathbb{R}^n)$ ; in particular,  $sp(H_1, \dots, H_k) = \overline{\tilde{J}(\mathbb{R}^n)} = \overline{J(\mathbb{R}^{2n})}$ .

Our aim is to decompose the family  $\{H_j\}_{j=1}^k$ , as we described before, i.e. constructing a suitable Borel measure over  $\overline{J(\mathbb{R}^{2n})}$ , a continuous field of Hilbert spaces over it and a unitary operator  $T$  from  $L^2(\mathbb{R}^n)$  to the corresponding direct integral, and satisfying the properties given in Theorem 2.2. Recall that such unitary operator should satisfies that  $[Tf(H_1, \dots, H_k)u](\lambda) = f(\lambda)[Tu](\lambda)$ . Motivated by the fact that  $[f(H_1, \dots, H_k)u](z) = f(\lambda)u(z)$  for every  $z \in \Sigma_\lambda$ , we will look for a continuous field structure over  $\{J(\mathbb{R}^{2n}) \ni \lambda \rightarrow L^2(\Sigma_\lambda, \eta_\lambda)\}$  and a unitary operator formally of the form  $Tu(\lambda) = (\alpha u)|_{\Sigma_\lambda}$ , where  $\eta_\lambda$  is the measure defined on  $\Sigma_\lambda$  given by its Riemannian structure and  $\alpha$  is some suitable function on  $\mathbb{R}^n$ . The function  $\alpha$  is placed there mainly to insure that  $T$  is unitary.

In what follows it will be useful to recall that  $J$  is regular at  $(x, \xi) \in \mathbb{R}^{2n}$  if the Jacobian  $D\tilde{J}(x) : \mathbb{R}^n \rightarrow \mathbb{R}^k$  has range  $k$  and this is equivalent to requiring that  $\wedge^k[D\tilde{J}(x)] : \wedge^k \mathbb{R}^n \rightarrow \wedge^k \mathbb{R}^k \cong \mathbb{R}$  is not identically 0.

Also let

$$\mathcal{I} = \left\{ \lambda \in \tilde{J}(\mathbb{R}^n) / \exists x \in \tilde{\Sigma}_\lambda \text{ such that } \wedge^k D\tilde{J}(x) = 0 \right\} \quad (4)$$

The well known Morse-Sard Theorem asserts that  $\mathcal{I}$  has null Lebesgue measure.

**Theorem 3.2.** *Assume that  $\mathcal{I}$  is closed. Let  $h \in C_c^\infty(\mathbb{R}^n)$ , then the map*

$$\tilde{J}(\mathbb{R}^n) \setminus \mathcal{I} \ni \lambda \rightarrow \int_{\tilde{\Sigma}_\lambda} h(z) d\eta_\lambda(z)$$

*is continuous with compact support. Let  $S$  be the set of vector fields  $u \in \prod_{\lambda \in J(\mathbb{R}^n) \setminus \mathcal{I}} L^2(\tilde{\Sigma}_\lambda, \eta_\lambda)$  such that, for each  $h \in C_c^\infty(\mathbb{R}^n)$ , the maps*

$$\tilde{J}(\mathbb{R}^n) \setminus \mathcal{I} \ni \lambda \rightarrow \|u(\lambda)\|_\lambda \quad \text{and}$$

$$\tilde{J}(\mathbb{R}^n) \setminus \mathcal{I} \ni \lambda \rightarrow \langle u(\lambda), h|_{\Sigma_\lambda} \rangle_\lambda$$

*are continuous. Then  $S$  makes  $\{J(\mathbb{R}^n) \setminus \mathcal{I} \ni \lambda \rightarrow L^2(\tilde{\Sigma}_\lambda, \eta_\lambda)\}$  a continuous field of Hilbert spaces.*

*Proof.* For the first part, it is enough to show continuity on each variable, for simplicity we shall do it on the first one. Lets fix  $\lambda_2^0, \dots, \lambda_k^0$ . For each  $\lambda_1 < \lambda_1'$  close enough, consider the Riemannian  $(n - k + 1)$ -manifold with boundary

$$N := \{(t, x) \in \mathbb{R} \times \mathbb{R}^n : \lambda_1 \leq t \leq \lambda_1', \tilde{J}(x) = (t, \lambda_2^0, \dots, \lambda_k^0) \text{ and } \wedge^k [D\tilde{J}(x)] \neq 0\}.$$

Then  $\partial N = \tilde{\Sigma}_{\lambda_1, \dots, \lambda_k^0} \cup \tilde{\Sigma}_{\lambda_1', \dots, \lambda_k^0}$ . The idea is to use the divergence theorem to express

$$\int_{\tilde{\Sigma}_{\lambda_1, \dots, \lambda_k^0}} h(x) d\eta_{\lambda_1, \dots, \lambda_k^0}(x) - \int_{\tilde{\Sigma}_{\lambda_1', \dots, \lambda_k^0}} h(x) d\eta_{\lambda_1', \dots, \lambda_k^0}(x)$$

as the integral over  $N$  of some function. We claim that the outgoing normal vector on  $\partial N$  can be extended over the whole  $N$ . Indeed,  $\wedge^k [DJ(x)] \neq 0$  is equivalent to say that there is an index  $I = \{i_1 < \dots < i_k\} \subset \{1, \dots, n\}$  such that  $\det(\partial_{i_j} \phi_m(x)) \neq 0$ , then each point  $(t, x) \in N$  belongs to a chart where  $\lambda$  is a variable (using the Implicit Function Theorem, one can locally express the variables indexed by such  $I$  as functions of the reminder  $n - k + 1$ ). This implies that at each point  $(t, x) \in N$  the tangent space  $T_{(t,x)}N$  has as subspace  $T_x \tilde{\Sigma}_{t, \lambda_2^0, \dots, \lambda_k^0}$ . We define  $n$  as the vector field defined by choosing a normal vector in  $T_{(t,x)}N$  orthogonal to  $T_x \tilde{\Sigma}_{t, \lambda_2^0, \dots, \lambda_k^0}$  considering orientation. Let  $X := hn$ ; then divergence theorem implies that the above expression is equal to

$$\int_N \operatorname{div}(X) v_N,$$

where  $v_N$  is the volume form of  $N$ . Moreover, it is easy to check that  $\operatorname{div}(X) = h \operatorname{div}(n) + \nabla h \cdot \nabla \phi_1$ , then  $\operatorname{supp}(\operatorname{div}(X)) \subseteq \operatorname{supp}(h)$ . Let  $I$  be an index as before and

$$U_I := \{(t, x) \in N : \det(\partial_{i_j} \phi_m(x)) \neq 0\}.$$

Clearly, the collection of all the  $U_I$ 's define a covering by charts of  $N$ . Note that, for each  $t$ , the restriction to  $\tilde{\Sigma}_{t, \lambda_2^0, \dots, \lambda_k^0}$  of each member of a partition of unity of  $N$  subordinated to the  $U_I$ 's, define a partition of unity of  $\tilde{\Sigma}_{t, \lambda_2^0, \dots, \lambda_k^0}$  subordinated to the covering by charts  $\{\tilde{\Sigma}_{t, \lambda_2^0, \dots, \lambda_k^0} \cap U_I : \{i_1 < \dots < i_k\} = I\}$ . Then

$$\int_N \operatorname{div}(X) v_N = \int_{\lambda_1}^{\lambda'_1} \int_{\Sigma_{t, \lambda_2^0, \dots, \lambda_k^0}} \operatorname{div}(X)(z) \left[ \frac{\det(g_{l,m}(t, z))}{\det(g_{l,m}^t(z))} \right]^{1/2} v_{\tilde{\Sigma}_{t, \lambda_2^0, \dots, \lambda_k^0}} dt,$$

where  $v_{\tilde{\Sigma}_{t, \lambda_2^0, \dots, \lambda_k^0}}$  is the volume form of  $\tilde{\Sigma}_{t, \lambda_2^0, \dots, \lambda_k^0}$ ,  $(g_{l,m}(t, z))$  and  $(g_{l,m}^t(z))$  are the metric matrices of  $N$  and  $\tilde{\Sigma}_{t, \lambda_2^0, \dots, \lambda_k^0}$  respectively (note that the quotient  $\frac{\det(g_{l,m}^t(z))}{\det(g_{l,m}(t, z))}$  is the first entry of  $(g^{l,m}(t, z)) = (g_{l,m}(t, z))^{-1}$ ). Then, absolute continuity of the integral implies the required continuity.

For the last part, note that condition i) in (2.4) follows by definition and condition ii) follows because  $C_c^\infty(\mathbb{R}^n) \subset S$ . Recall that for each open set  $O \subset \mathbb{R}^n$  and each compact  $K \subset O$  there is a smooth function  $b$  such that  $0 \leq b \leq 1$ ,  $b(K) = 1$  and  $b(O^c) = 0$  (a bump). For each  $q \in \mathbb{R}^n$  and  $0 < r < R$  fix a bump function  $b_{r,R}^q$  for  $O = B(q, R)$  and  $K = \overline{B(q, r)}$ . Then condition iii) follows clearly from considering the numerable collection  $\{b_{r,R}^q : r, R \in \mathbb{Q}, q \in \mathbb{Q}^n\} \subset S$ .  $\square$

**Remark 3.1.** If we wouldn't assume that  $\mathcal{I}$  is closed, almost the same argument, the Morse-Sard Theorem and the coarea formula (actually only the part concerning measurability, see below) would allow us to construct a measurable field of Hilbert spaces, but we prefer to insist on the more difficult to find continuous fields.

We will found  $\alpha$  after applying a classical result in geometric measure theory, the so called coarea formula: for any measurable function  $f$  on  $\mathbb{R}^n$ , the function  $\mathbb{R}^k \ni \lambda \rightarrow \int_{\tilde{\Sigma}_\lambda} f(z) d\eta_\lambda(z)$  is measurable, and we have that

$$\int_{\mathbb{R}^n} f(x) \|\wedge^k DJ(x)\| dx = \int_{\mathbb{R}^k} \int_{\tilde{\Sigma}_\lambda} f(z) d\eta_\lambda(z) d\lambda, \quad (5)$$

where the norm in the equation is the usual operator norm on linear maps.

This result can be found in [32] 10.6; it can also be easily deduced from Theorem 3.2.11 in [8]. It is stated using the  $(n - k)$ -Hausdorff measure on  $\mathbb{R}^n$  restricted to  $\tilde{\Sigma}_\lambda$ , but it coincide with the one coming from the Riemannian structure in our setting. When  $k = 1$ , it is easy to check that  $\|\wedge^k D\tilde{J}(x)\| = \|\nabla\phi(x)\|$ .

Coarea formula is one of the main ingredient of this article. In fact, note that a sort of coarea formula was already used in the proof of the first statement of the previous theorem (when we needed to compute an integral over  $N$ ).

We turn now to look for the required basic measure  $\eta$  on  $\overline{J(\mathbb{R}^{2n})}$ . By definition,  $A \subseteq \mathbb{R}^k$  is locally  $\eta$ -negligible iff  $A$  is  $\eta_u$ -negligible for every  $u \in L^2(\mathbb{R}^n)$ , where  $\eta_u$  is the measure defined by  $\eta_u(A) = \langle \chi_A(H_1, \dots, H_k)u, u \rangle$  for any Borel set  $A \subset \mathbb{R}^k$ . Then clearly,  $\eta$  is unique up to equivalence.

**Lemma 3.1.** *Assume that the set  $\{x \in \mathbb{R}^n / \wedge^k D\tilde{J}(x) = 0\}$  has null Lebesgue measure. Then  $\eta(A) = m(\tilde{J}(\mathbb{R}^n) \cap A)$ , where  $m$  is the Lebesgue measure.*

This result might seem unnecessary, because it seems quite likely that  $m(\overline{\tilde{J}(\mathbb{R}^n)} \setminus \tilde{J}(\mathbb{R}^n)) = 0$ . This is true for  $k = 1$ , but it already fails for  $k = 2$ . For instance, take a Jordan curve  $C$  in the plane with positive area, then the interior region is open and simply connected, its boundary is  $C$  and the Weierstrass uniformization Theorem implies it is biholomorphic with the open disc. So, just take  $\tilde{J}$  to be the composition of such holomorphic function with a smooth map from  $\mathbb{R}^n$  onto the open disc.

*Proof.* Note that, for every Borel set  $A \subset \mathbb{R}^k$ ,

$$\begin{aligned} \langle \chi_A(H_1, \dots, H_k)u, u \rangle &= \int_{\mathbb{R}^n} \chi_A(\tilde{J}(x))|u(x)|^2 dx = \\ &= \int_{\mathbb{R}^k} \int_{\tilde{\Sigma}_\lambda} \chi_A(\tilde{J}(z)) \|\wedge^k D\tilde{J}(z)\|^{-1} |u(z)|^2 d\eta_\lambda(z) d\lambda = \int_{A \cap \tilde{J}(\mathbb{R}^n)} \int_{\tilde{\Sigma}_\lambda} \|\wedge^k D\tilde{J}(z)\|^{-1} |u(z)|^2 d\eta_\lambda(z) d\lambda, \end{aligned}$$

then  $\eta_u(A) = 0$  for every  $u \in L^2(\mathbb{R}^n)$  iff  $m(\tilde{J}(\mathbb{R}^n) \cap A) = 0$ .  $\square$

The following theorem, even if it follows almost directly from coarea formula, we didn't find it stated elsewhere (except for the well known case  $k = 1$  and  $\phi(x) = \frac{x^2}{2}$ ).

**Theorem 3.3.** *Let  $\mathcal{M} = \{x \in \mathbb{R}^n / \wedge^k D\tilde{J}(x) \neq 0\}$  and define  $\rho : \mathcal{M} \rightarrow \mathbb{R}$  by  $\rho(x) = \|\wedge^k D\tilde{J}(x)\|^{-1}$ . Then the map*

$$T_x : L^2(\mathcal{M}) \rightarrow \bigoplus_{\overline{\tilde{J}(\mathbb{R}^n)}} L^2(\tilde{\Sigma}_\lambda, d\eta_\lambda) d\eta(\lambda),$$

given by

$$[T_x u(\lambda)](z) := \begin{cases} \rho(z)^{1/2} u(z) & \text{if } \lambda \in \tilde{J}(\mathbb{R}^n) \setminus \mathcal{I} \\ 0 & \text{otherwise.} \end{cases}$$

is unitary. If  $\mathcal{M}^c$  has null Lebesgue measure,  $T_x$  decompose the family  $\{\phi_j(Q)\}_{j=1}^k$ .

*Proof.* It follows from Sard's lemma and coarea formula that  $T$  is well defined and unitary. Moreover, for each function  $f$  Borel and bounded,  $z \in \tilde{\Sigma}_\lambda$  and  $u \in L^2(\mathbb{R}^n)$ , we have that

$$[T_x f(H_1, \dots, H_k) u(\lambda)](z) = \rho(z)^{1/2} [f(H_1, \dots, H_k) u](z) = \rho(z)^{1/2} f(\tilde{J}(z)) u(z) = f(\lambda) [T_x u(\lambda)](z).$$

□

**Remark 3.2.** This theorem would also follow under milder conditions over the family  $\{\phi_j\}_{j=1}$ , for instance it is enough to assume that  $\tilde{J}$  is Lipschitz on any bounded set.

We used the notation  $T_x$  for the unitary operator just to emphasize that it comes from a Hamiltonian which only depends on the position coordinates. Below we will give the correspondent results for momentum dependent hamiltonian and we will use the notation  $T_\xi$ .

If  $h_j(x, \xi) = \phi_j(\xi)$  everything follows almost in the same way. In this case  $\mathfrak{Op}(h_j) = \phi_j(P)$  is the convolution operator given by  $\phi_j$ . If we conjugate them with the Fourier transform  $\mathcal{F}$  we get the corresponding multiplication operator. Then  $\text{sp}(\phi_1(P), \dots, \phi_k(P)) = \overline{\tilde{J}(\mathbb{R}^n)}$  and to decompose them we only need to compose  $T_x$  with  $\mathcal{F}$ . On the classical side we only need to put a minus sign on the canonical symplectic form of  $T^*\tilde{\Sigma}_\lambda$ . Note also that the composition of the flows give us  $\Psi_{t_1, \dots, t_k}(x, \xi) = (x - t_1 \nabla \phi_1(\xi) - \dots - t_k \nabla \phi_k(\xi), \xi)$ . Summarizing:

**Theorem 3.4.** *Let  $\phi_j \in C^\infty(\mathbb{R}^n)$  and  $h_j(x, \xi) = \phi_j(\xi)$ .*

1. *Assume that the range of the matrix  $(\partial_l \phi_m(x))$  is  $k$  for every  $x \in \tilde{\Sigma}_\lambda$ , then the map*

$$\Sigma_\lambda \ni [(x, \xi)] \rightarrow (\xi, q_\xi(x)) \in T^*(\tilde{\Sigma}_\lambda) \quad (6)$$

*is a symplectomorphism, where  $[(x, \xi)]$  is the orbit of  $(x, \xi)$  by  $\Psi$ . Here  $T^*(\tilde{\Sigma}_\lambda)$  is endowed with minus the canonical symplectic structure on a cotangent bundle and  $T_\xi^*(\tilde{\Sigma}_\lambda)$  is identified as before with the  $(n - k)$ -dimensional plane subspace  $\langle \nabla \phi_1(\xi), \dots, \nabla \phi_k(\xi) \rangle^\perp$ .*

2. *Assume that the set  $\{x \in \mathbb{R}^n / \wedge^k D\tilde{J}(x) = 0\}$  has null Lebesgue measure, then*

$$T_\xi := T_x \circ \mathcal{F} : L^2(\mathbb{R}^n) \rightarrow \bigoplus_{\overline{\tilde{J}(\mathbb{R}^n)}} L^2(\tilde{\Sigma}_\lambda, dz_\lambda) d\eta(\lambda),$$

*decompose the family of operators  $\{\phi_j(P)\}_{j=1}^k$ .*

**Remark 3.3.** Later we are going to consider in more detail the important case  $k = 1$  and  $h(x, \xi) = |\xi|^2/2$ . For the moment, let's start by noticing some few facts. Clearly  $\nabla \phi(\xi) = \xi$ ,  $\mathcal{I} = \{0\}$  and  $\Sigma_\lambda = \mathbb{S}_{\sqrt{2\lambda}}^{n-1}$  is the  $n - 1$ -dimensional sphere of radius  $\sqrt{2\lambda}$  for each  $\lambda > 0$ . Also, in the quantum side, we get  $\phi(P) = -\frac{1}{2}\Delta$  is the Laplacian on  $\mathbb{R}^n$ . The decomposition of this operator is well known but usually it is presented using as constant fiber the Hilbert space  $L^2(\mathbb{S}^{n-1})$ , however in the definition of the corresponding  $T$  there is, at each  $\lambda > 0$ , the dilation factor  $\sqrt{2\lambda}$ , which gives a diffeomorphism between the 1-sphere and the  $\sqrt{2\lambda}$ -sphere. On the classical side, some elements in this theorem already appeared in the classical scattering theory literature. For instance,  $\{\xi\}^\perp$  is sometimes called the impact plane and  $\|q_\xi(x)\| = \|x - \frac{\langle x, \xi \rangle}{|\xi|^2} \xi\|$  is called the impact parameter.

Since in both general cases the required measurable field of Hilbert spaces comes from a continuous one, our claim state that  $T^*(\tilde{\Sigma}_\lambda)$  (either with  $+$  or  $-$  the standard symplectic structure) should be quantized into  $L^2(\tilde{\Sigma}_\lambda, \eta_\lambda)$ . Obviously, there is no need to recall our abstract ideas to claim this, because it is quite accepted that the phase space of a configuration space, i.e. its cotangent bundle, should be quantized into the  $L^2$ -space of the configuration space. So this cases are examples where our general claim follow.

## 4 Weyl-Landsman Quantization for the Cotangent Bundle of a Riemannian Manifold.

In this section we are going to give a solution for the problem of quantizing  $T^*(\tilde{\Sigma}_\lambda)$  into  $L^2(\tilde{\Sigma}_\lambda)$ . As we mention in the introduction, in [16] II.3 a quantization for the cotangent bundle of any Riemannian manifold was given.

Let  $(M, g)$  be a Riemannian manifold. Let  $\exp : U \subset TM \rightarrow M$  be the exponential map, i.e.  $U$  consist of all the points  $(m, X) \in TM$  for which a unique geodesic  $\gamma$  at  $m$  with initial velocity  $\gamma'(0) = X$  is defined at  $t = 1$ , and  $\exp(m, v) = \gamma(1)$ . It is well known that there is an open neighborhood  $V \subseteq U$  of the zero section in  $TM$  (identified with  $M$ ) such that the map  $\nu$  given by

$$V \ni (m, X) \rightarrow (\exp(m, \frac{1}{2}X), \exp(m, -\frac{1}{2}X)) \in M \times M$$

is a diffeomorphism with the image of  $V$ , which we denote by  $W$ .

Let

$$C_{PW}^\infty(T^*M) = \{f \in C^\infty(T^*M) / \hat{f} \in C_c^\infty(TM)\},$$

where

$$\hat{f}(m, v) = \int_{T_m^*M} f(m, \xi) e^{i\langle \xi, v \rangle} d\mu_m^*(\xi)$$

and

$$d\mu_m^*(\xi) = \frac{1}{(2\pi)^n \sqrt{\det g(m)}} d\xi$$

**Definition 4.1.** *The Weyl-Landsman quantization of  $f \in C_{PW}^\infty(T^*M)$  is given, for  $\hbar \neq 0$ , by the integral operator*

$$\mathfrak{Op}_\hbar^M(f)u(x) = \int_M K_\hbar[f](x, y)u(y)d\mu(y),$$

where

$$K_\hbar^\kappa[f](x, y) = \begin{cases} \hbar^{-\dim M} \kappa(\nu^{-1}(x, y)) \hat{f}(\nu^{-1}(x, y)/\hbar) & (x, y) \in W \\ 0 & \text{otherwise} \end{cases}$$

and  $\kappa$  is a smooth function on  $TM$  with the following properties:

- $\kappa = 1$  in a neighborhood  $V_\kappa \subset V$  of  $M$
- $\kappa$  has support in  $V$
- $\kappa(m, v) = \kappa(m, -v)$

**Remark 4.1.** Formally  $\nu^{-1}(x, y)$  gives the point at half the way between  $x$  and  $y$  and the velocity needed to return from there to  $x$  (which is the same needed to return to  $y$  but with opposite direction), just like in the usual Weyl Quantization.

The bump function  $\kappa$  is placed there just to insure that  $K_h^\kappa[f]$  is a smooth kernel with compact support, this was quite useful to prove strictness (Theorem III.3.5.1 in [16]). However we think that the procedure obtained by removing  $\kappa$  would have most of the properties of  $\mathfrak{Op}_h^M$ . Notice that in this way we still get a Hilbert-Schmidt operator. To study such procedure one should consider the kernel as a distribution (i.e. belonging to  $\mathfrak{D}'(M \times M)$ ) and look it as a limit of smooth kernels of the form  $K_h^\kappa[f]$  (for instance with increasing  $V_\kappa$ ). This perspective would also allow the possibility to enlarge the type of symbols we can work with, at first by considering the Schwartz's Kernel Theorem (elements in  $\mathfrak{D}'(M \times M)$  define operators from  $\mathfrak{D}(M) = C_c^\infty(M)$  to  $\mathfrak{D}'(M)$ ) and then imposing conditions to get pseudodifferential operators.

Moreover, Landsman noticed also in [16] that for each  $f \in C_{PW}^\infty(T^*M)$  fixed, there is  $\hbar_0$  such that  $\mathfrak{Op}_h^M(f)$  does not depend on  $\kappa$  for  $\hbar < \hbar_0$ , because for  $\hbar$  small enough,  $\hbar \text{supp}(\hat{f})$  is contained in  $V_\kappa$  and this implies we can remove  $\kappa$  from the expression of  $\mathfrak{Op}_h^M(f)$ . In particular any semiclassical results do not depend on  $\kappa$ .

Weyl-Landsman quantization was defined also over to the following type of symbols:

Let  $a \in C^\infty(M)$ , if we consider it as a function on  $T^*M$  (independent of the cotangent variable) then  $\mathfrak{Op}_h^M(a)$  are just the multiplication operator given by  $a$ .

Also, let  $X$  be a complete vector field on  $M$  and consider the smooth function given by  $J_X(m, \xi) = \langle X(m), \xi \rangle_m$ , where  $\langle \cdot, \cdot \rangle_m$  denote the duality between the tangent and the cotangent space at the point  $m$ . Then

$$\mathfrak{Op}_h^M(J_X) = -i\hbar(X + \frac{1}{2}\text{div}(X)), \quad (7)$$

where  $\text{div}X$  is the divergence of  $X$ . See proposition II.3.6.1 in [16].

In both cases the corresponding operators are essentially selfadjoint in  $C_c^\infty(M)$ .

It is known how to compute the one parameter groups corresponding to those operators (propositions II 3.6.2 and II 3.6.4). Clearly  $e^{it\mathfrak{Op}_h^M(a)}u(z) = e^{ita(z)}u(z)$ . We also have that

$$e^{\frac{it}{\hbar}\mathfrak{Op}_h^M(J_X)}u(z) = \sqrt{\frac{d[(\Phi_{-t}^X)^*v]}{dv}}(z)u(\Phi_{-t}^X(z)), \quad (8)$$

where  $v$  is the volume form on  $M$  defined by its Riemannian structure,  $\Phi_t^X$  is the flow corresponding to  $X$  and  $\frac{d[(\Phi_{-t}^X)^*v]}{dv}$  is the Radon-Nikodym derivative of the measure corresponding to  $(\Phi_{-t}^X)^*v$  respect to the measure corresponding to  $v$ .

The computations done in [16] to obtain the expression of  $\mathfrak{Op}_h^M(J_X)$  given above, also allow us to define  $\mathfrak{Op}_h^M$  over finite products of symbols of the form  $J_X$ , with  $X$  complete. This will be useful in the next section. To do it, recall the proof of proposition II.3.6.1 in [16]. There an expression of  $\mathfrak{Op}_h^M(f)$  was given using normal coordinates. It was noticed there that, if  $f$  becomes a polynomial in the momentum variable on such coordinates, then the expression obtained has sense as an oscillatory integral. This is the case for  $f = J_X$  and for finite products of such symbols as well. In fact, we can compute such integrals explicitly, as in the  $f = J_X$  case, we would only need to use that the Fourier transform of the polynomial  $\xi^\alpha$  as a distribution is  $(i)^{|\alpha|}D^\alpha$  times the Dirac delta (using standard notation), but we will not need to do such computations in this article. We postpone to study such operators in a forthcoming work.

The commutation relations satisfied by hamiltonians independent of momentum or of the form  $J_X$  can be found in the table (4), as well as the ones satisfied by the operators corresponding to them through Weyl-Landsman quantization.

C.C.R	Q.C.R.
$\{a, b\} = 0$	$\frac{1}{\hbar} [\mathfrak{Op}_\hbar^M(a), \mathfrak{Op}_\hbar^M(b)] = 0$
$\{J_X, a\} = Xa$	$\frac{1}{\hbar} [\mathfrak{Op}_\hbar^M(J_X), \mathfrak{Op}_\hbar^M(a)] = \mathfrak{Op}_\hbar^M(Xa) = Xa$
$\{J_X, J_Y\} = J_{[X, Y]}$	$\frac{1}{\hbar} [\mathfrak{Op}_\hbar^M(J_X), \mathfrak{Op}_\hbar^M(J_Y)] = \mathfrak{Op}_\hbar^M(J_{[X, Y]}).$

Table 1: Commutation Relations

## 5 a decomposable Weyl Calculus and Commutation of Quantization and Reduction

Let us come back to our initial quantization problem. From now on we will use Weyl-Landsman calculus to quantize  $T^*\tilde{\Sigma}_\lambda$  into  $L^2(\tilde{\Sigma}_\lambda)$  and, whenever there is no ambiguity, we will use the notation  $\mathfrak{Op}_\hbar^\lambda$  instead of  $\mathfrak{Op}_\hbar^{\tilde{\Sigma}_\lambda}$ . Also, on this section, we will always assume that the set  $\{x \in \mathbb{R}^n / \wedge^k D\tilde{J}(x) = 0\}$  has null Lebesgue measure.

Recall that we arrived to this point from a Marsden-Weinstein reduction started from the phase space  $\mathbb{R}^{2n}$  in the classical side, and a decomposition of  $L^2(\mathbb{R}^n)$  as a direct integral in the quantum side; this processes come from the family of Hamiltonians  $h_j(x, \xi) = \phi_j(x)$  (or  $h_j(x, \xi) = \phi_j(\xi)$ ), and the corresponding quantized family  $H_j = \mathfrak{Op}(h_j)$  respectively, where  $\mathfrak{Op}$  is the Weyl calculus. We are now in the position to introduce one the main novelties of this article: a new calculus. Let  $f \in C^\infty(\mathbb{R}^{2n})$  such that  $\{h_j, f\} = 0$  for every  $1 \leq j \leq k$ . This implies that  $f \circ \Phi_t = f \forall t \in \mathbb{R}^k$ ; so for each  $\lambda \in \tilde{J}(\mathbb{R}^n) \setminus \mathcal{I}$ , we can consider the smooth function  $f^\lambda$  on the orbit space  $\Sigma_\lambda \cong T^*\tilde{\Sigma}_\lambda$ , given by  $f^\lambda([x, \xi]) = f(x, \xi)$ , where  $[x, \xi]$  denote the orbit of  $(x, \xi)$  by  $\Phi$ . Then, if  $f^\lambda$  is an admissible symbol for  $\mathfrak{Op}_\hbar^\lambda$  for almost every  $\lambda$ , we can essentially define the field of operators  $\{\tilde{J}(\mathbb{R}^n) \ni \lambda \rightarrow \mathfrak{Op}_\hbar^\lambda(f^\lambda)\}$ . Therefore we can consider the operator on the Hilbert space  $\int_{\tilde{J}(\mathbb{R}^n)}^\oplus L^2(\tilde{\Sigma}_\lambda) d\lambda$  denoted by  $\int_{\tilde{J}(\mathbb{R}^n)}^\oplus \mathfrak{Op}_\hbar^\lambda(f^\lambda) d\lambda$  and given by

$$\left[ \int_{\tilde{J}(\mathbb{R}^n)}^\oplus \mathfrak{Op}_\hbar^\lambda(f^\lambda) d\lambda \right] (\{u\})(\lambda_0) = \mathfrak{Op}_\hbar^{\lambda_0}(f^{\lambda_0})(\{u\})(\lambda_0),$$

where  $\{u\}$  denote a suitable element of the direct integral. The domain of this operator is the subspace of the direct integral where the above expression has sense and define an element of the direct integral. This is a sort of decomposable operator (we don't require the family to be essentially uniformly bounded).

**Definition 5.1.** Let  $\mathcal{A}^d$  be the space of functions  $f \in C^\infty(\mathbb{R}^{2n})$  such that  $\{h_j, f\} = 0$  for all  $j = 1, \dots, n$  and  $f^\lambda$  is an admissible symbol for  $\mathfrak{Op}_\hbar^\lambda$  for almost every  $\lambda$ . For each  $f \in \mathcal{A}$ , we define

$$\mathfrak{Op}_\hbar^d(f) := T^* \left[ \int_{\tilde{J}(\mathbb{R}^n)}^\oplus \mathfrak{Op}_\hbar^\lambda(f^\lambda) d\lambda \right] T.$$

We call  $\mathfrak{Op}_h^d$  the  $(H_1, \dots, H_k)$ -decomposable Weyl calculus and  $\mathcal{A}^d$  the space of admissible symbols for  $\mathfrak{Op}_h^d$ . We denote by  $\mathcal{A}_1^d$  the space of symbols  $f \in \mathcal{A}^d$  such that  $f^\lambda \in C_{PW}^\infty(\Sigma_\lambda)$  for almost all  $\lambda$ .

The definition of the space of admissible symbols for  $\mathfrak{Op}_h^d$  depends of the scope of Weyl-Landsman quantization, so if this get extended, the space  $\mathcal{A}^d$  could become larger. Below we will give some elements of this space. Also note that clearly  $\mathcal{A}_1^d$  is a Poisson algebra.

We will not study  $\mathfrak{Op}^d$  properly in this article, instead our purpose here is to show some reasons why this could be useful and interesting. We will show some properties it has in relation with the well known Weyl calculus  $\mathfrak{Op}$ .

We want to consider the following question: Does reduction commute with quantization in this context? Lets explain the meaning of this question. Since  $\{h_j, f\} = 0$ , one might expect (naively) that  $\mathfrak{Op}_h(h_j)$  and  $\mathfrak{Op}_h(f)$  also commute. Proposition (2.2) implies that, if  $\int_{\tilde{J}(\mathbb{R}^n)}^\oplus \mathfrak{Op}^\lambda(f^\lambda) d\lambda$  is bounded, then it commutes with each  $T\mathfrak{Op}(h_j)T^*$ . So one could expect that  $\mathfrak{Op}_h^d(f)$  is equal to  $\mathfrak{Op}_h(f)$ . Such equality is what we call (strong) commutation of quantization and reduction.

**Definition 5.2.** Let  $f \in \mathcal{A}$  be a symbol such that  $\mathfrak{Op}_h(f)$  define a closed symmetric operator for each  $h \neq 0$  in a neighborhood of 0. Assume that there is a common core  $D$  for each  $\mathfrak{Op}_h(f)$ , such that  $D \subseteq \text{Dom}(\mathfrak{Op}^d(f))$ .

- We say that quantization and reduction commute on  $f$  strongly if for each  $u \in D$

$$\mathfrak{Op}_h(f)u = \mathfrak{Op}_h^d(f)u.$$

- We say that quantization and reduction commute on  $f$  semiclassically if for each  $u \in D$

$$\lim_{h \rightarrow 0} (\mathfrak{Op}_h(f) - \mathfrak{Op}_h^d(f))u = 0.$$

We denote by  $\mathcal{A}_s$  the set of all symbols on which quantization and reduction commute semiclassically.

**Remark 5.1.** Note that if  $Tu(\lambda) \in \text{Dom}(\mathfrak{Op}_h^\lambda(f^\lambda))$  for  $\eta$ -almost every  $\lambda$ , then quantization and reduction commute on  $f$  strongly iff

$$[T\mathfrak{Op}_h(f)u](\lambda) = \mathfrak{Op}_h^\lambda(f^\lambda)[Tu(\lambda)], \quad (9)$$

for almost all  $\lambda$ .

**Remark 5.2.** Recall that the Groenewold-van Hove Theorem [10, 33] (or for example Theorem 4.59 in [9]) implies that the equality  $\frac{i}{h}[\mathfrak{Op}_h(g), \mathfrak{Op}_h(f)] = \mathfrak{Op}_h(\{g, f\})$  fails to be true in general. Actually the proof is constructive and it seems to suggest that the inequality happen for a large set of symbols. Because of this, we think that there is not reason to expect that reduction and quantization commute strongly on a large set of symbols neither (but we will give below an interesting class of examples). However, we do expect it happen semiclassically.

**Remark 5.3.** Recall that, in general the meaning and properties of  $\mathfrak{Op}_h(f)$  depends on what “type” of symbol is  $f$ . For instance, if  $f$  belongs to  $S_{\rho, \delta}^m(\mathbb{R}^{2n})$  (see definition 1.1[31]), then  $\mathfrak{Op}_h(f)$  is a bounded linear operator between the test functions space  $C_c^\infty(\mathbb{R}^n)$  and the space  $C^\infty(\mathbb{R}^n)$  and it can be extended to a bounded linear operator between  $\mathcal{E}'(\mathbb{R}^n)$  and  $\mathcal{D}'(\mathbb{R}^n)$ . Under extra conditions, one could get that  $\mathfrak{Op}_h(f)$  define a selfadjoint operator; for example, Theorem 26.2 in [31] states



that, if  $f$  is a hypoelliptic symbol (definition 25.1 in [31]), then  $\mathfrak{Op}_h(f)$  is essentially selfadjoint on  $C_c^\infty(\mathbb{R}^n)$ . Also, the Calderón-Vaillancourt theorem assert that, if  $f \in BC^\infty(\mathbb{R}^{2n})$ , then  $\mathfrak{Op}_h(f)$  is bounded.

We should study  $\mathfrak{Op}^d$  on that direction. For instance, it is not difficult to propose a distributional definition of  $\mathfrak{Op}_h^d$ , the natural candidate is

$$\langle \mathfrak{Op}_h^d(f)u, \psi \rangle := \int_{\tilde{J}(\mathbb{R}^n)} \langle \mathfrak{Op}_h^\lambda(f^\lambda)[Tu(\lambda)], T\psi(\lambda) \rangle d\lambda$$

Note that if  $\psi \in C_c^\infty(\mathbb{R}^n)$  then  $T\psi(\lambda) \in C_c^\infty(\Sigma_\lambda)$ , for each  $\lambda \in \tilde{J}(\mathbb{R}^n) \setminus \mathcal{I}$ , and the above expression could make sense even if  $\mathfrak{Op}_h^\lambda(f^\lambda)[Tu(\lambda)]$  is a distribution. The only issue would be that the right hand might not be well defined. However, the scope of Weyl-Landsman calculus hasn't been developed enough to make such notion worth yet. If that is the case a suitable definition of the different versions of commutation of reduction and quantization should be given.

Concerning selfadjointness, we expect we can check it for symbols that are finite product of symbols of the form  $J_Y$ , with  $Y$  complete, at least.

We also expect that, if  $f \in \mathcal{A}_1^d$  then  $\mathfrak{Op}^d(f)$  is bounded (or at least this should hold on an interesting subalgebra). This would be a first step to prove strictness of  $\mathfrak{Op}^d$  on a suitable Poisson algebra. Meanwhile, we will impose some conditions to get a weak versions of such result.

**Proposition 5.1.** *Let  $f, g \in \mathcal{A}_1^d$  such that  $\mathfrak{Op}_h^d(f)$ ,  $\mathfrak{Op}_h^d(g)$ ,  $\mathfrak{Op}_h^d(fg)$  and  $\mathfrak{Op}_h^d(\{f, g\})$  are uniformly bounded in  $\hbar$  (belonging to some neighborhood of 0). Then, for each  $u \in L^2(\mathbb{R}^n)$ ,*

$$\lim_{\hbar \rightarrow 0} \left( \mathfrak{Op}_h^d(f \cdot g) - \mathfrak{Op}_h^d(f) \star \mathfrak{Op}_h^d(g) \right) u = 0$$

and

$$\lim_{\hbar \rightarrow 0} \left( \mathfrak{Op}_h^d(\{f, g\}) - \frac{i}{\hbar} [\mathfrak{Op}_h^d(f), \mathfrak{Op}_h^d(g)] \right) u = 0,$$

where  $A \star B = \frac{1}{2}(AB + BA)$  and  $[A, B] = AB - BA$ .

*Proof.* Both equalities follow from the dominated convergence theorem and the strictness of each  $\mathfrak{Op}^\lambda$  (theorem III.3.5.1 [16]).  $\square$

**Remark 5.4.** To get strictness one must get such equalities not only strongly but in the operator norm sense. Those are called von Neumann's and Dirac's conditions. The last requirement is Rieffel's condition:  $\lim_{\hbar \rightarrow 0} \|\mathfrak{Op}_h^d(f)\| = \|f\|_\infty$ . We cannot use the same ideas to get it, because we cannot intertwine limits and supremum if we have only pointwise convergence. This issues suggest two angles to approach this problem: estimate  $\|\mathfrak{Op}_h^d(f)\|$ , and try to look for conditions to insure that the field of operators  $\{\tilde{J}(\mathbb{R}^n) \ni \lambda \rightarrow \mathfrak{Op}_h^\lambda(f^\lambda)\}$  is actually continuous ( recall we constructed the direct integral involved from a continuous field of Hilbert spaces).

Under similar conditions we can start to build a Poisson algebra in  $\mathcal{A}_s$ .

**Proposition 5.2.** *Let  $f, g \in BC^\infty(\mathbb{R}^{2n}) \cap \mathcal{A}_s$  such that  $\mathfrak{Op}_h^d(f)$ ,  $\mathfrak{Op}_h^d(g)$  and  $\mathfrak{Op}_h^d(fg)$  are uniformly bounded in  $\hbar$ . Then  $fg \in \mathcal{A}_s$ . The same holds if we replace  $fg$  for  $\{f, g\}$ .*

*Proof.*

$$[\mathfrak{Op}_h(fg) - \mathfrak{Op}_h^d(fg)]u =$$

$$[\mathfrak{Op}_h(f.g) - \mathfrak{Op}_h(f) \star \mathfrak{Op}_h(g)]u + [\mathfrak{Op}_h(f) \star \mathfrak{Op}_h(g) - \mathfrak{Op}_h^d(f) \star \mathfrak{Op}_h^d(g)]u + \\ + [\mathfrak{Op}_h^d(f) \star \mathfrak{Op}_h^d(g) - \mathfrak{Op}_h^d(fg)]u.$$

Since the Weyl Calculus is strict on  $BC^\infty(\mathbb{R}^{2n})$ , the first term vanish when  $\hbar \rightarrow 0$ . It is clear that  $\star$  is strongly continuous, so the second term also vanish. The previous proposition implies that the last term vanish too.

After replacing  $\star$  by  $\frac{i}{\hbar}[\cdot, \cdot]$ , the same arguments work to prove the second claim.  $\square$

Lets look for symbols where quantization and reduction commute strongly on. If  $h_j(x, \xi) = \phi_j(x)$  (resp.  $h_j(x, \xi) = \phi_j(\xi)$ ) and  $f(x, \xi) = a(x)$  for some measurable function  $a$  (resp.  $f(x, \xi) = a(\xi)$ ), then clearly quantization and reduction commute strongly on  $f$ .

Now we will show some not so obvious examples of such symbols. For simplicity, we are going to work out explicitly only the case  $h_j(x, \xi) = \phi_j(x)$ ; the other case follows from this one.

Let  $Y$  be a complete vector field on  $\mathbb{R}^n$  such that  $Y(x) \perp \nabla \phi_j(x)$ ,  $\forall j = 1, \dots, k \forall x \in \mathbb{R}^n$ . In such case we say that  $Y$  is tangent to each  $\tilde{\Sigma}_\lambda$ . It is easy to check that  $\{h_j, J_Y\} = 0 \forall j$ . We are going to prove that quantization and reduction commute strongly on  $J_Y$ . Note that, we can define on each  $\tilde{\Sigma}_\lambda$  the vector field  $Y^\lambda(z) := (i_*^\lambda)^{-1}(Y(z))$ . It is straightforward to show that  $J_Y^\lambda = J_{Y^\lambda}$ . Therefore, in order to compare the operators involved, we need to compute  $\text{div} Y^\lambda$ . Recall that we defined  $\rho(x) = \|\wedge^k D\tilde{J}(x)\|^{-1}$ .

**Proposition 5.3.** *Let  $Y$  be a vector field on  $\mathbb{R}^n$  tangent to each  $\tilde{\Sigma}_\lambda$ . Also let  $\lambda_0 \in \tilde{J}(\mathbb{R}^n) \setminus \mathcal{I}$  and  $Y^{\lambda_0}$  be the vector field on  $\tilde{\Sigma}_{\lambda_0}$  given by  $Y^{\lambda_0}(z) := (i_*^{\lambda_0})^{-1}(Y(z))$ . Then, for each  $z \in \Sigma_{\lambda_0}$ ,*

$$\text{div}(Y)(z) = \text{div}(Y^{\lambda_0})(z) - \rho^{-1}(z)Y(\rho)(z) = \text{div}(Y^{\lambda_0})(z) - 2\rho^{-1/2}(z)Y(\rho^{1/2})(z).$$

*Proof.* The last equality follows from the chain rule. In order to compare both divergences, we will decompose pointwise the canonical volume form  $dx$  of  $\mathbb{R}^n$  using the volume form  $v_\lambda$  of  $\tilde{\Sigma}_\lambda$ . More precisely, we will look for an expression of the form  $dx = v_\lambda \wedge v_N$ . But we would need to compute  $v_N$ . It will be easier to do this in the diffeomorphic manifold  $M := \text{graf}(\tilde{J}) := \{(x, \lambda) \in \mathbb{R}^n \times \mathbb{R}^k : \tilde{J}(x) = \lambda\}$ .

For simplicity, assume that  $\det(\partial_{n-k+j}\phi^m(x)) \neq 0$ . Recall that the implicit function theorem allow us to take as a chart for  $M$  the map  $\Psi(x_1, \dots, x_n, \lambda) = (x_1, \dots, x_{n-k}, \lambda)$ . Note that, if we fix  $\lambda \in \mathbb{R}^k$ , the maps  $\Psi^\lambda(x) = (x_1, \dots, x_{n-k})$  define a chart on  $\tilde{\Sigma}_\lambda$ . Let  $\{\frac{\partial}{\partial z_j}|_{(x, \lambda)}, \frac{\partial}{\partial \mu_m}|_{(x, \lambda)}\}$  and  $\{dz_j|_{(x, \lambda)}, d\mu_m|_{(x, \lambda)}\}$ , with  $j = 1, \dots, n-k$  and  $m = 1, \dots, k$ , be the corresponding bases of  $T_{(x, \lambda)}M$  and  $T_{(x, \lambda)}^*M$  respectively.

Endowing  $M$  with the volume form  $v_M = \pi^*(dx)$ , where  $\pi : M \rightarrow \mathbb{R}^n$  is the projection in the  $\mathbb{R}^n$ -component, and using the coarea formula, we get that, for every  $f \in C_c^\infty(M)$ ,

$$\int_M f v_M = \int_{\mathbb{R}^n} f(x, \tilde{J}(x)) dx = \int_{\mathbb{R}^k} \left[ \int_{\tilde{\Sigma}_\lambda} f(x, \lambda) \rho(x) v_\lambda \right] d\lambda.$$

Therefore  $v_M = v_\lambda \wedge (\rho d\mu)$  locally (note that  $d\lambda$  is not necessarily equal to  $d\mu$ ). Recall that the divergence of a vector field  $X$  over a manifold with a volume form  $w$  can be defined by the equation  $\text{div}(X)w = d(i_X(w))$ , where  $d$  is the exterior derivative and  $i_X(w)$  is the interior product of  $w$  by  $X$ . Since  $Y$  is tangent to each  $\tilde{\Sigma}_\lambda$ , we have that  $i_Y(\rho d\mu) = 0$ . Then,

$$i_Y(v_M) = i_Y(v_\lambda \wedge \rho d\mu) = i_Y(v_\lambda) \wedge \rho d\mu + v_\lambda \wedge i_Y(\rho d\mu) = i_Y(v_\lambda) \wedge \rho d\mu.$$

Its clear that  $d(i_Y(v_\lambda)) \wedge \rho d\mu = \text{div}(Y^\lambda)v_M$ , therefore

$$\text{div}(Y)v_M = d(i_Y(v_\lambda) \wedge \rho d\mu) = \text{div}(Y^\lambda)v_M + (-1)^{n-k-1}i_Y(v_\lambda) \wedge d(\rho d\mu).$$

Lets compute the last term. Let  $\rho_0 \in C^\infty(M)$  such that  $v_\lambda = \rho_0 dz_1 \wedge \cdots \wedge dz_{n-k}$ . Also let  $\{Y_j(x)\}_{j=1}^{n-k}$  be the coordinates of  $Y(x)$  in the base  $\{\frac{\partial}{\partial z_j}|_x\}$ . Since  $i_Y(v_\lambda)(X_1, \cdots X_{n-k-1}) = v_\lambda(Y, X_1, \cdots X_{n-k-1})$ , where  $X_1, \cdots, X_{n-k-1}$  are vector fields, we have that  $i_Y(v_\lambda) = \sum_j a_j dz_1 \wedge \cdots \wedge \hat{dz}_j \cdots \wedge dz_{n-k}$ , where  $a_j = v_\lambda(Y, \frac{\partial}{\partial z_1}, \cdots, \frac{\partial}{\partial z_j}, \cdots, \frac{\partial}{\partial z_{n-k}})$ . Using the explicit expression of  $v_\lambda$  above and the properties of the determinant, we get  $a_j = (-1)^j Y_j$ . On the other hand  $d(\rho d\mu) = \sum_j \frac{\partial \rho}{\partial z_j} dz_j \wedge d\mu$ . Therefore

$$\begin{aligned} i_Y(v_\lambda) \wedge d(\rho d\mu) &= \sum_j (-1)^j \rho_0 Y_j dz_1 \wedge \cdots \wedge \hat{dz}_j \cdots \wedge dz_{n-k} \wedge \sum_j \frac{\partial \rho}{\partial z_j} dz_j \wedge d\mu = \\ &= \left[ \sum_j (-1)^{n-k} Y_j \frac{\partial \rho}{\partial z_j} \right] v_\lambda \wedge d\mu = (-1)^{n-k} \rho^{-1} Y(\rho) v_M \end{aligned}$$

□

**Remark 5.5.** The idea of decomposing the volume of  $\mathbb{R}^n$  as the exterior product of the volume form of  $\tilde{\Sigma}_\lambda$  and a complement form comes from foliated cohomology theory, for instance [7].

When  $k = 1$ , a direct computation shows that:

**Corollary 5.1.** *Let  $Y$  be a vector field on  $\mathbb{R}^n$  such that  $Y(x) \perp \nabla \phi(x)$  for each  $x \in \mathbb{R}^n$ . Let  $\lambda \in \tilde{J}(\mathbb{R}^n) \setminus \mathcal{I}$  and  $Y^\lambda$  be the vector field on  $\tilde{\Sigma}_\lambda$  given by  $Y^\lambda(z) := (i_*^\lambda)^{-1}(Y(z))$ . Then, for each  $z \in \Sigma_\lambda$ ,*

$$\text{div} Y^\lambda(z) = \text{div} Y(z) + \frac{1}{\|\nabla \phi(z)\|^2} \langle \text{Hess}[\phi](z) Y(z), \nabla \phi(z) \rangle,$$

where  $\text{Hess}[\phi](z)$  is the Hessian matrix of  $\phi$  at  $z$ .

**Remark 5.6.** If  $\phi(x) = x^2/2$ , then  $\text{Hess}[\phi](x)$  is the identity matrix for every  $x$  and, since we are assuming that  $\langle Y(x), \nabla \phi(x) \rangle = 0$ , we get that  $\text{div} Y(z) = \text{div} Y^\lambda(z)$  for each  $z \in \tilde{\Sigma}_\lambda$ .

**Theorem 5.1.** *Let  $Y$  be a complete vector field on  $\mathbb{R}^n$  tangent to each  $\tilde{\Sigma}_\lambda$ . Then quantization and reduction commute on  $J_Y$  strongly.*

*Proof.* Let  $z \in \mathbb{R}^n$  regular (respect to  $\tilde{J}$ ). Then, using the same notation than above, we obtain

$$\begin{aligned} &\left( T_x^* \left[ \int_{\tilde{J}(\mathbb{R}^n)}^\oplus -i\hbar(Y^\lambda + \frac{1}{2}\text{div} Y^\lambda) d\lambda \right] T_x u \right) (z) = \\ &= -\rho(z)^{-\frac{1}{2}} i\hbar(Y^{\tilde{J}(z)} + \frac{1}{2}\text{div} Y^{\tilde{J}(z)})(\rho^{\frac{1}{2}} u|_{\tilde{\Sigma}_{J(z)}})(z) = \\ &= -i\hbar Y u(z) - i\hbar \rho(z)^{-1/2} Y(\rho^{1/2})(z) u(z) - \frac{i\hbar}{2} (\text{div} Y(z) + 2\rho(z)^{-1/2} Y(\rho^{1/2})(z) u(z)) = \\ &= \left[ -i\hbar(Y + \frac{1}{2}\text{div} Y) u \right] (z). \end{aligned}$$

□

**Corollary 5.2.** *Let  $Y$  be a complete vector field on  $\mathbb{R}^n$  tangent to each  $\Sigma_\lambda$  and let  $\Phi_t^Y$  its corresponding flow. Then, if  $\lambda_0 \in \tilde{J}(\mathbb{R}^n) \setminus \mathcal{I}$  and  $z \in \tilde{\Sigma}_{\lambda_0}$ ,*

$$\frac{d[(\Phi_t^Y)^* m]}{dm}(z) = \rho^{-1}(z)[\rho \circ \Phi_t^Y](z) \frac{d[(\Phi_t^Y)^* v_{\lambda_0}]}{dv_{\lambda_0}}(z).$$

Also, for each function  $f$  on  $\mathbb{R}^n$  Borel measurable,

$$\int_{\mathbb{R}^n} f(\rho^{-1} \circ \Phi_t^Y)[(\Phi_t^Y)^* m] = \int_{\tilde{J}(\mathbb{R}^n)} \int_{\tilde{\Sigma}_\lambda} f[(\Phi_t^Y)^* v_\lambda] d\lambda.$$

*Proof.* Since reduction and quantization commute on  $J_Y$ , we have that

$$e^{\frac{it}{\hbar} \mathfrak{Op}_\hbar(J_Y)} = T_x^* \left( \int_{\tilde{J}(\mathbb{R}^n)}^{\oplus} e^{\frac{it}{\hbar} \mathfrak{Op}_\hbar^\lambda(J_Y^\lambda)} d\lambda \right) T_x.$$

The first equality follows from this and the explicit computation of  $e^{\frac{it}{\hbar} \mathfrak{Op}_\hbar^M(J_X)}$  given in (8). The second equality follows after applying coarea formula.  $\square$

**Remark 5.7.** Unfortunately we have not found yet interesting examples to apply this results. Note that the last equation is a generalization of coarea formula. We think that requiring  $Y$  to be complete is not necessary, this result should follow under milder conditions, but this is not the purpose of this article.

**Proposition 5.4.** *Let  $\mathcal{A}_0$  be the Poisson algebra generated by smooths functions independent of the cotangent component and those of the form  $J_Y$ , with  $Y$  complete and tangent to each  $\tilde{\Sigma}_\lambda$ . Then every element of  $\mathcal{A}_0$  is of the form*

$$a + \sum_j^m J_{Y_{j_1}} \cdots J_{Y_{j_l}}$$

where  $a \in C^\infty(\tilde{\Sigma}_\lambda)$  and each vector field involved is complete and tangent to each  $\Sigma_\lambda$ .

*Proof.* Recall that  $\{a, b\} = 0, \{J_Y, a\} = Y(a)$  and  $J_Y, J_Z = J_{[Y, Z]}$ . It is clear that if  $Y, Z$  are complete vector fields tangents to each  $\Sigma_\lambda$ , so it is  $[Y, Z]$ . Also  $aJ_Y = J_{aY}$ . So, using the properties of the Poisson bracket, we get our claim.  $\square$

Since position dependent symbols and those of the form  $J_Y$ , with  $Y$  complete and tangent to each  $\Sigma_\lambda$ , belong to  $\mathcal{A}_s$ , we could expect that the same happen for finite products of them, so the previous results would suggest that  $\mathcal{A}_0 \subset \mathcal{A}_s$ , but for the moment, we don't have the tools to prove it. For instance we cannot use the arguments in the proof of proposition 5.2 because each  $\mathfrak{Op}_\hbar^d(J_Y)$  is unbounded.

Actually we don't know neither if quantization and reduction commute strongly on  $\mathcal{A}_0$ . We thought at first this isn't true and it would be enough to prove it that the image of some element of  $\mathcal{A}_0$  by  $\mathfrak{Op}_\hbar$  do not formally commute with each  $H_j$ , but we have tried with some powers of some  $J_Y$ 's and they still commute with each  $H_j$ . In the next section we will give some details about this.

Finally let us describe a setting on which interesting Poisson algebras become candidates to be in  $\mathcal{A}_s$ . Lets  $G$  be a Lie group acting on  $\mathbb{R}^n$ . Such action induce a Lie algebra homomorphism  $\zeta : \mathfrak{g} \rightarrow \mathfrak{X}(\mathbb{R}^n)$  given by

$$\zeta(v)(x) = \frac{d}{dt} (exp(tv) \cdot x) |_{t=0},$$

where  $\mathfrak{g}$  is the Lie algebra corresponding to  $G$ . Assume that  $\zeta(v)$  is complete and tangent to each  $\tilde{\Sigma}_\lambda \cong \mathbb{S}_{\sqrt{\lambda}}^{n-1}$ .

Let  $J_v := J_{\zeta(v)}$  (or  $\langle J, v \rangle$  in the standard notation of symplectic geometry), then the map  $\mathfrak{so}(n) \ni v \rightarrow J_v \in C^\infty(T^*\mathbb{R}^n)$  is also a Lie algebra homomorphism. Actually, this is a restriction of a Poisson map  $j : C^\infty(\mathfrak{g}^*) \rightarrow C^\infty(T^*\mathbb{R}^n)$  constructed as follows. Consider the moment map  $\mathcal{J} : T^*\mathbb{R}^n \rightarrow \mathfrak{g}^*$  given by

$$\mathcal{J}[(x, \xi)](v) = \langle \xi, \zeta(v)(x) \rangle,$$

where  $\langle \cdot, \cdot \rangle$  denotes the duality between  $T_x\mathbb{R}^n$  and  $T_x^*\mathbb{R}^n$ . If we endow  $\mathfrak{g}^*$  with minus the Lie-Poisson structure, then  $J$  is a Poisson map.

Note that, for each  $v \in \mathfrak{g}$ , if we denote by  $E_v$  the linear functional on  $\mathfrak{g}^*$  given by evaluating on  $v$ , then  $E_v \in C^\infty(\mathfrak{g}^*)$  and  $\mathcal{J}^*(E_v) = J_v$ , where  $\mathcal{J}^*$  is the pullback of  $\mathcal{J}$ . So, we can take  $j = \mathcal{J}^*$ . Moreover

**Proposition 5.5.** *Let  $\mathcal{J}$  be the moment map considered above. Then  $\{h_j, \mathcal{J}^*(a)\} = 0$ , for each  $a \in C^\infty(\mathfrak{g}^*)$ .*

*Proof.* Note that, for each  $t \in \mathbb{R}^k$  and  $(x, \xi) \in T^*\mathbb{R}^n$ ,

$$\begin{aligned} [\mathcal{J} \circ \Phi_t(x, \xi)](v) &= [\mathcal{J}(x, \sum_j^k t_j \nabla \phi_j(x) + \xi)](v) = \langle \zeta(v)(x), \sum_j^k t_j \nabla \phi_j(x) + \xi \rangle = \\ &= \langle \zeta(v)(x), \xi \rangle = \mathcal{J}(x, \xi)(v). \end{aligned}$$

Then

$$\mathcal{J}^*(a) \circ \Phi_t = a \circ \mathcal{J} \circ \Phi_t = a \circ \mathcal{J} = \mathcal{J}^*(a).$$

□

So, under those assumptions we have that  $\mathcal{J}^*[C^\infty(\mathfrak{g}^*)] \subset \mathcal{A}$ . Then, the Poisson algebra generated by  $C^\infty(\mathbb{R}^n)$  and  $\mathcal{J}^*[C^\infty(\mathfrak{g}^*)]$  is also a subset of  $\mathcal{A}$ . This is a large space where we expect that quantization and reduction commute, at least semiclassically. We will give an example of this setting in the next section.

We pretend to continue developing this subject, but we decided to postpone it to treat it separately in other article. For example, we expect that, once Weyl-Landsman quantization get extended enough, quantization and reduction will commute semiclassically on a large subalgebra of  $\mathcal{A}$ . One of purposes to do so is the following: if quantization and reduction commute semiclassically on  $f$ , we propose to interpret  $\mathfrak{Op}_h^d(f)$  as an effective Hamiltonian for  $\mathfrak{Op}_h(f)$ . This is a formal claim, there is no definition of the notion of effective Hamiltonian, but based on the well known examples, one would require that at least it must be a decomposable operator for a suitable direct integral (up to unitary equivalence). Of course this is the case for  $\mathfrak{Op}_h^d(f)$ .

## 6 An important example: $h(x, \xi) = \|\xi\|^2$ , $O(n)$ symmetries and angular momenta.

When  $k = 1$  and  $h(x, \xi) = \|\xi\|^2$ , the geometry of  $\tilde{\Sigma}_\lambda = \mathbb{S}_{\sqrt{\lambda}}^{n-1}$  (the  $(n-1)$ -sphere of radius  $\sqrt{\lambda}$ ) allow us to compute explicitly Weyl-Landsman quantization and to find examples of vector fields tangents to each  $\tilde{\Sigma}_\lambda$ .

Let start by writing more explicitly our quantization. First note that for each  $\lambda \geq 0$ ,

$$V \equiv V_\lambda = \{(z, \eta) \in T\mathbb{S}_{\sqrt{\lambda}}^{n-1} \mid \|\eta\|_z < \frac{\sqrt{\lambda}\pi}{2}\},$$

$$W \equiv W_\lambda = \{(z, w) \in \mathbb{S}_{\sqrt{\lambda}}^{n-1} \times \mathbb{S}_{\sqrt{\lambda}}^{n-1} \mid z \neq -w\}$$

and

$$\nu^{-1}(z, w) = \begin{cases} (\frac{\sqrt{\lambda}}{\|z+w\|}(z+w), \frac{\sqrt{\lambda}\theta}{2\|z-w\|}(z-w)) & \text{if } z \neq w \\ (z, 0) & \text{if } z = w. \end{cases}$$

where we identify  $T_z\mathbb{S}_{\sqrt{\lambda}}^{n-1}$  with  $z^\perp$  and  $\theta$  is the angle between  $z$  and  $w$  ( $\cos \theta = \frac{\langle z, w \rangle}{\lambda}$ ).

In what follows, just to simplify the expressions, we will not include the bump function  $\kappa$  in the definition of  $\mathfrak{Op}_h^\lambda$ . We should mention though this can be done using a bump function with support in the  $(n-1)$ -disc of radius  $\frac{\sqrt{\lambda}\pi}{2}$  and such that  $\kappa \circ \nu^{-1}(z, w) = \kappa^1(\theta)$ , where  $\kappa^1$  has support in  $[0, \pi]$  and it is equal to 1 on an interval  $[0, \theta_0]$ . Then

$$[\mathfrak{Op}_h^\lambda(f)u](z) = \hbar^{n-1} \int_{\mathbb{S}_{\sqrt{\lambda}}^{n-1} - \{-z\}} \hat{f}(\frac{\sqrt{\lambda}(z+w)}{\|z+w\|}, \frac{\sqrt{\lambda}\theta}{2\hbar} \frac{(z-w)}{\|z-w\|}) u(w) d\mu(w). \quad (10)$$

From here it seems clear that choosing stereographic projection to study  $\mathfrak{Op}_h^\lambda$  could be useful. So, let us fix some notation concerning this. For each  $w \in \mathbb{S}_{\sqrt{\lambda}}^{n-1}$  let fix an ordered orthonormal base  $\{w_j\}_{j=1}^n$  with  $w_n = w/\sqrt{\lambda}$ . We define the stereographic projection with north pole at  $w$  by

$$\Psi_w^\lambda : \mathbb{S}_{\sqrt{\lambda}}^{n-1} - \{w\} \rightarrow \{w\}^\perp$$

$$\Psi_w^\lambda(z) = \frac{\lambda}{\lambda - \langle z, w \rangle} (z - \frac{\langle z, w \rangle}{\lambda} w) = \frac{\lambda}{\langle z, w \rangle + \lambda} P_{\{w\}^\perp}(z),$$

where  $P_{\{w\}^\perp}(z)$  is the orthogonal projection on the subspace orthogonal to  $w$ . If we identify  $\{w\}^\perp$  with  $\mathbb{R}^{n-1}$  using our orthonormal base, we obtain the stereographic coordinates

$$z_j := \langle \Psi_w^\lambda(z), w_j \rangle = \frac{\lambda \langle z, w_j \rangle}{\langle z, w \rangle + \lambda},$$

In this way we get a chart for  $\mathbb{S}_{\sqrt{\lambda}}^{n-1}$  defined on  $\mathbb{S}_{\sqrt{\lambda}}^{n-1} - \{w\}$ . The inverse of this stereographic chart is given by

$$\Upsilon_w^\lambda(z_1, \dots, z_{n-1}) = \frac{\lambda}{\|\hat{z}\|^2 + \lambda} (\hat{z} - w) + w$$

where  $\hat{z} := \sum_{j=1}^{n-1} z_j w_j$ .

Let  $\{\frac{\partial}{\partial z_j}|_z\}_{j=1}^{n-1}$  be the corresponding base of the tangent space at  $z \in \mathbb{S}_{\sqrt{\lambda}}^{n-1} - \{w\}$ . It is easy to check that  $[i^\lambda]_* (\frac{\partial}{\partial z_j}|_w) = 2w_j$ . Moreover,

$$\langle [i^\lambda]_* (\frac{\partial}{\partial z_j}|_z), [i^\lambda]_* (\frac{\partial}{\partial z_k}|_z) \rangle = \frac{4\lambda^2 \delta_{jk}}{(\lambda + |z|^2)^2},$$

then  $\{\frac{\partial}{\partial z_j}|_z\}$  is a orthogonal base of  $T_z\mathbb{S}_\lambda^{n-1}$  and

$$\sqrt{\det g^\lambda(z)} = \left( \frac{2\lambda}{(\lambda + |z|^2)^2} \right)^{n-1}.$$

Taking as north pole  $-z$ , we get another expression for our quantization

$$(2\lambda\hbar)^{1-n}[\mathfrak{Op}_\hbar^\lambda(f)u](z) = \int_{\mathbb{R}^{n-1}} \hat{f}\left(\frac{\sqrt{\lambda}(z + \hat{x})}{\|z + \hat{x}\|}, \frac{\sqrt{\lambda}}{\hbar} \arccos\left(\frac{\|x\|}{\sqrt{\|x\|^2 + \lambda}}\right) \frac{z - \Upsilon_z^\lambda(x)}{\|z - \Upsilon_z^\lambda(x)\|}\right) u(\Upsilon_z^\lambda(x)) \left[\frac{1}{\lambda^2 + \|x\|^2}\right]^{2(n-1)} dx.$$

where  $\hat{x} = \sum_{j=1}^{n-1} x_j w_j$ .

The canonical action of  $O(n)$  on  $\mathbb{R}^n$  clearly satisfies the conditions required at the end of the last section. Then each element of  $\mathcal{J}^*[C^\infty(\mathfrak{so}(n)^*)]$  Poisson commutes with  $\|x\|^2$ . In particular, each power of  $J_v$ , with  $v \in \mathfrak{so}(n)$ , Poisson commutes with  $\|x\|^2$ . We thought at first that quantization and reduction don't commute strongly on some of those symbols, and it would be enough to show it that their images by  $\mathfrak{Op}_\hbar$  do not formally commute with  $-\Delta$ ; but this is not the case for the important examples we will give now (until the forth power).

Let  $f_{ij}(x, \xi) = x_i \xi_j - x_j \xi_i$ , with  $i \neq j$ . It is easy to check that  $f_{ij} = J_v$ , where  $v$  is an infinitesimal generator of a rotation of the plane generated by the elements of the canonical base  $e_i$  and  $e_j$ , and leaving the rest fixed. Actually  $\zeta(v) = x_i \frac{\partial}{\partial x_j} - x_j \frac{\partial}{\partial x_i}$ . The  $f_{ij}$ 's are usually called angular momenta.

Direct computations shows that:  $\mathfrak{Op}_\hbar(f_{ij}^2) = \hbar^2/2 + \mathfrak{Op}_\hbar(f_{ij})^2$ ,  $\mathfrak{Op}_\hbar(f_{ij}^3) = \mathfrak{Op}_\hbar(f_{ij})^3 + 2\hbar^2 \mathfrak{Op}_\hbar(f_{ij})$  and  $\mathfrak{Op}_\hbar(f_{ij}^4) = \mathfrak{Op}_\hbar(f_{ij})^4 + 5\hbar^2 \mathfrak{Op}_\hbar(f_{ij})^2 + \frac{3}{2}\hbar^4$ . Therefore  $[\Delta, \mathfrak{Op}_\hbar(f_{ij}^m)] = 0$ , for  $m = 1, 2, 3, 4$  (computed in  $C_c^\infty(\mathbb{R}^n)$ ). Because of this, we expect that quantization and reduction commute strongly on  $f_{ij}^m$ , for  $m = 1, 2, 3, 4$ .

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